References

Aizenman, Warzel

Resonant Delocalization for Random Schroedinger Operators on Tree Graphs

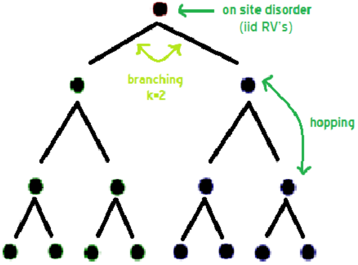
J.Eur.Math.Soc. 15, 1167-1222 (2013)

Extended States in Lifshitz Tail Regime for Random Schroedinger Operators on Trees

Phys.Rev.Lett. 106, 136804 (2011)

The Setting

The setting: Anderson on Trees



Non interacting particles in random medium:

$$H_\lambda = \lambda \sum_x \epsilon_x n_x + \sum_{<x,y>} \left(c_x^\dagger c_y + c_y^\dagger c_x \right)$$

$$\left[H_\lambda \psi(x) = -\sum_{y \sim x} \psi(y) + \lambda \sum_x V(x) \psi(x) \right]$$

Dynamical Question:

Long time behaviour of initially localized wave function $\psi_0(y) = \delta_{x,y}$?

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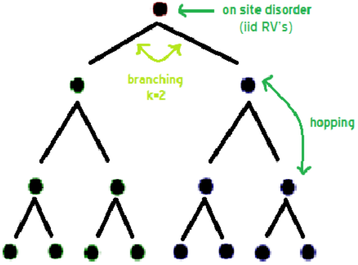

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Dynamical Question:

Long time behaviour of initially localized wave function $\psi_0(y) = \delta_{x,y}$?

$$\Longrightarrow G_{xy}^\lambda(t) = -i\theta(t) \langle x | e^{-iH_\lambda t} | y \rangle$$

Relevant quantity to look at

Retarded Green function:

$$G^{\lambda}_{xy}(E+i\eta) = \int_{-\infty}^{\infty} dt G^{\lambda}_{xy}(t) e^{i(E+i\eta)t} = \langle x | \frac{1}{E+i\eta-H_{\lambda}} | y \rangle$$

Dynamical properties captured by:

$$\lim_{\eta\to 0} \Im\text{mm}\, G^{\lambda}_{xx}(E+i\eta) = \begin{cases} \text{smooth} & \text{DELOCALIZATION} \\ \text{deltas} & \text{LOCALIZATION} \end{cases}$$

WHY?

Nonzero immaginary part of local self energy implies decay of local excitations

$$G^{R}_{xx}(E+i\eta) = \int d\xi \frac{\rho_x(\xi)}{E+i\eta-\xi} = \frac{1}{E+i\eta-\epsilon_x-\Sigma_x(E+i\eta)}$$

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Dynamical properties captured by:

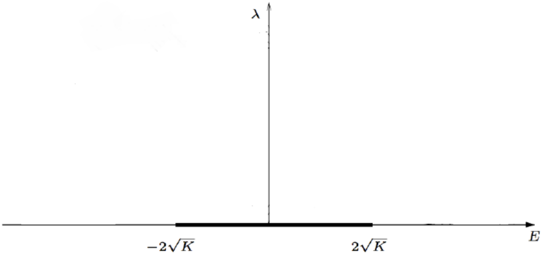
$$\lim_{\eta\to 0} \mathbb{P}\left(\mathfrak{Imm}\left(G^{\lambda}_{xx}(E+i\eta)\right) > 0 \right) = \begin{cases} > 0 & \text{DELOCALIZATION} \\ = 0 & \text{LOCALIZATION} \end{cases}$$

WHY?

Nonzero immaginary part of local self energy implies decay of local excitations

$$G^{R}_{xx}(E+i\eta) = \int d\xi \frac{\rho_x(\xi)}{E+i\eta-\xi} = \frac{1}{E+i\eta-\epsilon_x-\Sigma_x(E+i\eta)}$$

The phase diagram and open questions



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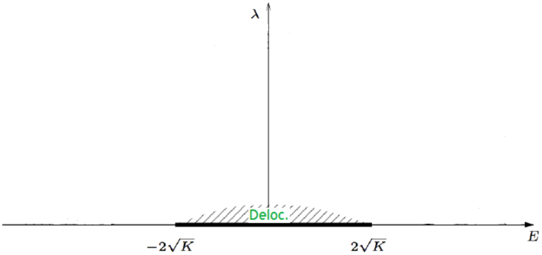
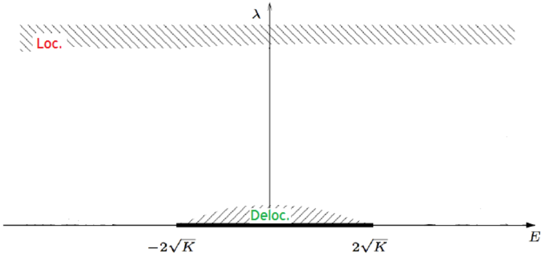


- Delocalization in pert. regime: Klein (1998), Miller-Deridda (1993)

The phase diagram and open questions



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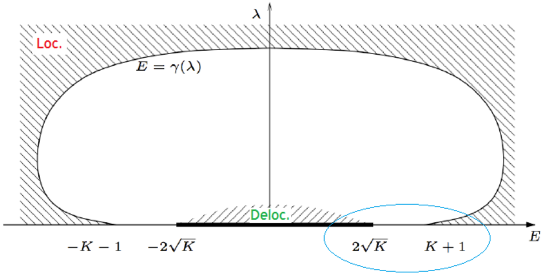

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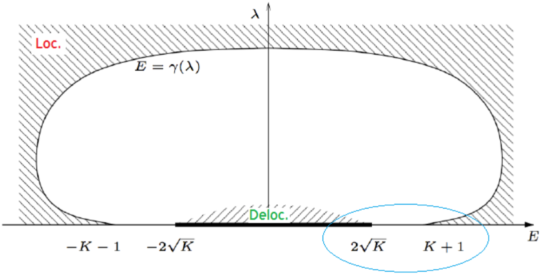

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- Localization at strong disorder: Abou Chacra-Anderson-Thouless (1973), Aizenman (1994), Aizenmann-Molchanov (1993)
- Close to band edges: Miller-Deridda (1993), Abou Chacra-Thouless (1974)

Open Questions:

1. How to characterize mobility edge?
2. Nature of the spectrum at the band edges, i.e. location of mobility edge at small disorder?

The answers (paper's spoiler)

How to characterize mobility edge?

Whenever either:

$$L_\lambda(E) := -\mathbb{E}\left(\log\left|G^\lambda_{00}(E+i0)\right|\right) < \log k$$

or:

$$\phi_\lambda(1,E) := \lim_{s\to 1}\left[\lim_{|x|\to\infty}\frac{\log\mathbb{E}\left(\left|G^\lambda_{\mathbf{0}x}(E+i0)\right|^s\right)}{|x|}\right] > -\log k$$

then with probability one $\Im\mathrm{mm}\, G^\lambda_{00}(E+i0) > 0$

DELOCALIZATION OCCURS!

Nature of the spectrum at the band edges?

There is delocalization. It is due to a geometric, nonperturbative mechanism involving resonances.

Outline of the Talk

Characterization of the mobility edge

- What this geometry allows
- Make sense of the criteria
- Proof: relevance of resonances

Delocalization at the band edges

- Why is it surprising
- Why is it possible

Characterization of Mobility Edge

Why this geometry?

Theory: many body localization Altshuler, Gefen, Kamenev, Levitov (1997)
Practice: can handle Green functions: recursion and factorization.

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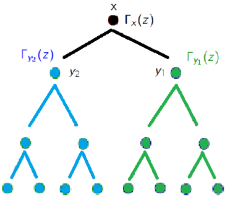
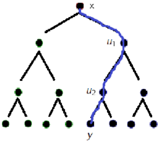


$$G_{xx}(z) = \left(z - \lambda\epsilon_x - \sum_{y\in\mathcal{N}_x} \Gamma_y^{(-)}(z)\right)^{-1}$$

$\Gamma_u^{(-)}(z)$ Green function in subtree rooted at u



$$G_{xy}(z) = G_{xx}(z) \prod_{0<u\leq y} \Gamma_u^{(-)}(z)$$
$$= G_{yy}(z) \prod_{0<u\leq y} \Gamma_u^{(+)}(z)$$

(Abou Chacra, Anderson, Thouless)

$$\mathfrak{Imm}\left(G_{xx}(E+i\eta)\right) = |G_{xx}(z)|^2 \left[\eta + \sum_{y \in \mathcal{N}_x} \mathfrak{Imm}\left(\Gamma_y(z)\right)\right]$$

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Iterate from the root of tree:

$$\mathfrak{Imm}\left(\Gamma_0(E+i\eta)\right) \geq \sum_{|x|=n} \Big| \underbrace{\prod_{0\leq u\leq x} \Gamma_u(z)}_{\text{weigth of a path}} \Big|^2 \sum_{y\in\mathcal{N}_x} \mathfrak{Imm}\left(\Gamma_y(E+i\eta)\right)$$

sum over all paths
from root to generation x

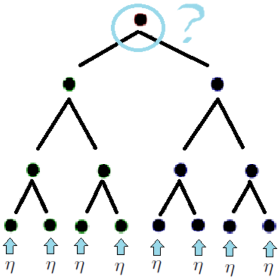
weigth of a path

Some complicated directed polymer:

$$\mathcal{Z}_n(z) = \sum_{x:|x|=n} \Big| \prod_{0\leq u\leq x} \Gamma_u(z) \Big|^2 = \sum_{x:|x|=n} |G_{0x}(z)|^2 = \sum_{\substack{paths\\|p|=n}} \left(\omega^{(n)}_{path}\right)^2$$

How to exploit it (were they numbers):

$$\Im m \Gamma_0(z) \geq \sum_{x:|x|=n} \left(\prod_{0 \leq u \leq x} |\Gamma_u(z)|^2 \sum_{y \in \mathcal{N}_x} \Im m \Gamma_y(z) \right)$$



1. Perturb fixed point solution $\Im m \Gamma(z) = 0$ with infinitesimal η

   infinitesimally weak coupling to a dissipative bath

2. Need to control the asymptotics of:

$$\mathcal{Z}_n(z) = \sum_{x \in S_n} \prod_{0 \leq u \leq x} |\Gamma_u(z)|^2 \sim z^n$$

Transition when $\log \mathcal{Z}_n(z)$ changes sign

Local excitations couple to bath on sites infinite far away: deloc!

(Abou Chacra, Anderson, Thouless)

$$\mathfrak{Imm}\,(G_{xx}(E+i\eta)) = |G_{xx}(z)|^2 \left[\eta + \sum_{y\in\mathcal{N}_x} \mathfrak{Imm}\,(\Gamma_y(z))\right]$$

Iterate from the root of tree:

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sum over all paths
from root to generation x

weigth of a path

Identically distributed random variables!

Some complicated directed polymer:

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How to exploit it (for RV):

What does '$\mathbb{Z}_n(z)$ asymptotically big' mean?

How precisely it implies delocalization?

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$$\lim_{\eta\to 0} \mathbb{P}\left(\max_{x\in S_n} \prod_{0\leq u\leq x} |\Gamma_u(z)|^2 \geq e^{\delta n} \Big| \mathcal{A}_x \right) \geq p_0 > 0$$

weight of optimal path

for all n big enough, $\mathcal{A}_x = 1_{\max_{y\in\mathcal{N}^+_{\S}} \mathcal{I}\mathbf{mm}\Gamma(y,z)\geq\xi(\alpha,z)}$ and $\xi(\alpha,z)$ is the $\alpha-$percentile.

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How it implies delocalization?

1. Assume $\lim_{\eta\to 0} \mathbb{P}\left(\Im m\Gamma(E + i\eta) > 0\right) = 0$.

2. Show it is not possible when $\mathcal{Z}_n(z)$ asymptotically grows with n: *contraddiction of continuity property of the prob.distributions when* $\eta \to 0$

Up to now:

Restate the problem of delocalization on trees into the problem of the asymptotic distribution of free energy of a polymer.

Delocalization whenever the free energy is negative with positive probability.

Make sense of criterion I

$$L_\lambda(E) := -\mathbb{E}\left(\log\left|G^\lambda_{00}(E+i0)\right|\right) < \log k$$

What they prove for you:

Typical asymptotic decay of Green function (for $\eta \in (0,\eta_0)$, ϵ arbitrary):

$$\lim_{|x|\to\infty} \mathbb{P}\left(|G^\lambda_{\mathbf{0}x}(E+i\eta)| \in e^{-L(E)|x|}\left[e^{-\epsilon|x|}, e^{\epsilon|x|}\right]\right) = 1$$

The sequence of 'free energies' $\frac{\log|G^\lambda_{\mathbf{0}x}(E+i\eta)|}{|x|}$ converges to the number $L_\lambda(E)$
would be a quenched free energy in the DirectedPolymer (Derrida-Spohn).

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What you are asking with criterion?

The limiting value ('total' quenched free energy) is negative.

Make sense of criterion II

$$\phi_\lambda(s,E)\Big|_{s=1} = \lim_{s\to 1}\left[\lim_{|x|\to\infty}\frac{\log\mathbb{E}\left(\left|G^\lambda_{0x}(E+i0)\right|^s\right)}{|x|}\right] > -\log k$$

What they prove for you:

Large deviation estimate for the sequence of free energies (for $|x| = n$ big and $\eta\in(0,\eta_0)$):

$$\mathbb{P}\left(|G^\lambda_{0x}(E+i\eta)| \in e^{-\gamma|x|}\left[e^{-\epsilon|x|}, e^{\epsilon|x|}\right]\right) \approx e^{-I(\gamma)|x|}$$

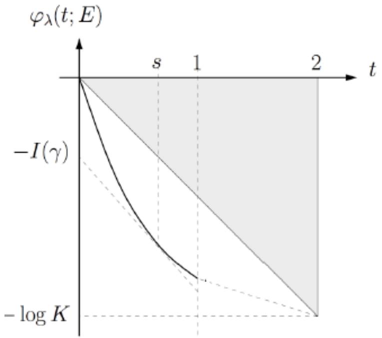
$$\text{with } I(\gamma) = -\inf_{s\in(0,1)}\left[\phi_\lambda(s,E)+s\gamma\right]$$

What are you asking?

The 'total' annealed free energy $\log k + \log\mathbb{E}\left[\omega^{(n)}_{path}\right] \sim \log\int d\gamma e^{-\gamma n}e^{-nI(\gamma)}$ becomes negative at the localization transition

Why this requires work?

Need to show that you *can* use large deviation when $\eta \to 0$: control $\phi_\lambda(s,z)$ in this limit: can be done for $s \in (0,1)$.



Why is this criterion stronger?

It holds: $\left.\frac{\partial\phi_\lambda(s,E)}{\partial s}\right|_{s=0} = -L_\lambda(E).$

(*use convexity of* ϕ)

Up to now:

Restate the problem of delocalization on trees into the problem of the asymptotic distribution of free energy of a polymer.

Delocalization whenever the free energy is negative with positive probability.

Criteria are given in terms of typical value of the free energies and large deviations.

Inside the proof: Resonances

When do paths with arbitrarily big weight occur for all big n?

$$\lim_{\eta \to 0} \mathbb{P}\left(\max_{\substack{paths \\ |p|=n}} \omega_{path}^{(n)} > e^{\delta n} \right)$$

(under hypothesis $\lim_{\eta \to 0} \Im mm G(z) = 0$)

Inside the proof: Resonances

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(under hypothesis $\lim_{\eta \to 0} \mathfrak{Imm} G(z) = 0$)

Claim:

The relevant mechanism are resonances at a given site x: local random energy ϵ_x and the local self energy $\Sigma_x(E)$ are resonating.

$$\begin{cases} |G_{xx}(z)| \geq e^{(\gamma+\delta)|x|} \\ |G_{0x^-}(z)| \geq e^{-\gamma|x|} \end{cases} \implies |G_{0x}(z)| \geq e^{\delta|x|}$$

for some $\gamma > 0$ (free parameter)

These events are extremely rare:

$$\mathbb{P}\left(|G_{xx}(z)| \geq e^{(\gamma+\delta)|x|}\right) \geq e^{-(\gamma+\delta)|x|}$$

$$\mathbb{P}\left(|G_{0x^-}(z)| \geq e^{-\gamma|x|}\right) = \begin{cases} \approx 1 & \text{if } \gamma = L_\lambda(E) \\ \approx e^{-I(\gamma)|x|} & \text{if } \gamma < L_\lambda(E) \end{cases}$$

one resonance here, the other on the moon....

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How can they be relevant at all?

We are on a tree: exponentially growing volume! How many of them do I expect at a given distance n:

$$\mathbb{E}\left[N(n)\right] = \begin{cases} \approx k^n e^{-(L_\lambda+\delta)n} & \text{if } \gamma = L_\lambda(E) \\ \approx k^n e^{-(I(\gamma)+\gamma+\delta)n} & \text{if } \gamma < L_\lambda(E) \end{cases}$$

In the remaining 20 pages?

Prove that resonances occur *regularly*: $\mathbb{P}\left(N(n) \geq 1\right) \geq \frac{\mathbb{E}[N]^2}{\mathbb{E}[N^2]}$

control probability of joint non independent events, substitute estimates with inequalities, take care of η

Up to now:

Restate the problem of delocalization on trees into the problem of the asymptotic distribution of free energy of a polymer.

Delocalization whenever the free energy is negative with positive probability.

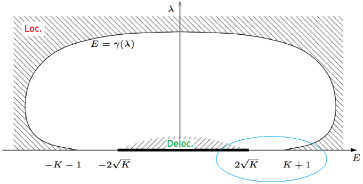
Criteria are given in terms of typical value of the free energies and large deviations.

Estimate the probability of negative free energy by restricting to rare resonances.

Their probability of occurrence decays exponentially, but they may be found to occur at all distance scales if the phase space grows exponentially fast.

Delocalized Spectrum at Small Disorder

Extract info without computations



There are delocalized states there!

$$L_\lambda(E) := -\mathbb{E}\left(\log\left|G_{00}^{\lambda}(E+i0)\right|\right) < \log k$$

Compute the Lyapunov exponent at zero disorder:

$$L_0(E) = \begin{cases} \frac{1}{2}\log k & \text{if } |E| \leq 2\sqrt{k} \\ \in \left(\frac{1}{2}\log k, \log k\right) & \text{if } 2\sqrt{k} < |E| < k+1 \\ \geq \log k & \text{if } |E| \geq k+1 \end{cases}$$

Show it is continuous in λ (*'the average over intervals of energies'*)

How do extended states form

Why is this result a surprise?

The density of states there is extremely small (*'Lifschitz tails'*):

$$N_\lambda(E) \leq e^{-\frac{C(E)}{\lambda^2}} \text{ for } E < -2\sqrt{k}$$

States are very rare, you expect them to be localized in isolated wells.

This is what happens in finite d (Kirsch (2008), Pastur-Figotin (1992))

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Why instead delocalized states on tree?

These states locally appear localized, exponentially small tunneling amplitudes ($e^{-L_\lambda(E)R}$): probability of mixing with another state at a *specified location* exponentially small.

But exponentially many channels: $\min \Delta E \approx \left[\rho^\lambda_{DOS}(E) K^R\right]^{-1}$.

'formation of extended states through fluctuation-enabled resonances between states that up to a certain scale may appear to be localized'.

Conclusions

- On trees, necessary conditions for delocalization involving typical decay and large deviation functions for Green functions;
- By continuity argument, prove delocalization in Lifschitz tail regime;
- Delocalized states in this regime form by hybridization of local 'quasimodes'. Allowed by exponential growth of volume.