Interacting extension of the Aubry–André model

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Statistical Physics PhD Course

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Summary

○ Introduction: localization and Aubry–André model
○ Interacting extension
  - the problem and the numerical approximation
  - numerical results: behaviour of a few quantities
  - detecting the MBL transition
  - tentative phase diagram
○ Conclusions
Let's consider systems with on site disorder.

**Localization:**

**one particle:**
localized wavefunctions (exponentially decaying envelope)

**many particles:**
breaking of ergodicity (local quantities have localized correlators)

Representative model: Anderson model (fermions)

\[
H = \sum_{\alpha} \xi_{\alpha} c_{\alpha}^{\dagger} c_{\alpha} - t \sum_{\{\alpha,\beta\}} (c_{\beta}^{\dagger} c_{\alpha} + c_{\alpha}^{\dagger} c_{\beta})
\]

- \(\xi_{\alpha}\): on site random energies
- \(t\): hopping
Introduction

Let’s consider systems with on site disorder.

**Localization:**

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<tr>
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- on site random energies
- hopping

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Interacting AA model
disorder is not necessary: 1D Aubry–André (AA) model

\[ H = \sum_{i=0}^{L-1} \left[ \xi_i c_i^\dagger c_i - t (c_i^\dagger c_{i+1} + c_{i+1}^\dagger c_i) \right] \]

\[ \xi_i = W \cos(2\pi ki + \delta), \quad k : \text{quasiperiodic potential} \]

peculiarity: no mobility edge; transition point at \( \frac{t}{W} = \frac{1}{2} \)
Let’s now consider many particles and turn on interaction.

\[ H = \sum_{i=0}^{L-1} \left[ \xi_i n_i + t(c_i^\dagger c_{i+1} + c_{i+1}^\dagger c_i) + \Delta n_i n_{i+1} \right] \]

\[ \xi_i = W \cos(2\pi k_i + \delta) \] (as in the AA model)

What happens with regards to localization?

S. Iyer, G. Refael, V. Oganesyan, D. Huse

**Notation:** \( g \equiv t/W, \ u \equiv \Delta/W, \) transition for \( g = \frac{1}{2} \) (at \( \Delta = 0 \))
Many body localization

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Many-Body Localization in a Quasiperiodic System,


S. Iyer, G. Refael, V. Oganesyan, D. Huse

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The approach:
Take a random element of the configurations space and let it evolve. Does the system thermalize?

End result:
the interacting model has a MBL transition. Interactions favours the delocalized phase.

Numerically simpler: sequentially hop on each bond

\[ H_m = H_{\text{on site}} + H_{\text{int}} + Lt(c_{m+1}^\dagger c_m + c_m^\dagger c_{m+1}) \]

\[ U(\Delta t) = \prod_{m=0}^{L-1} U_m(\Delta t), \quad \text{with } U_m(\Delta t) = \exp\left( -\frac{iH_m\Delta t}{L} \right) \]

It’s a different model: introduces new transitions; \( \Delta t \) must be small, or else qualitatively altered dynamics!
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Notes on the numerics:

- Free boundary conditions: no hopping over the boundary
- Quasiperiodic potential $W \cos(2\pi ki + \delta)$ with $k = \phi^{-1}$, $\delta$ RV
- Initial configuration: random in half filling configuration space

Lattice sizes are (empirically) chosen to minimize finite size effects: $L = 8 \div 20$, even.

Quantities to look at:

- temporal autocorrelator and its average (over sites and samples)
  \[ \chi_j(t) = (2\langle n_j \rangle(t) - 1)(2\langle n_j \rangle(0) - 1) \]
- participation ratio ($V$ is the size of configuration space)
  \[ \eta = \frac{1}{[IPR/V]}, \quad IPR = \langle \sum_c |\psi_c|^4 \rangle_{\text{smp}} \]
- Rényi entanglement entropy
  \[ S_2(t, L) = \langle -\log_2 \left( \text{tr}_A [\rho_A^2(t)] \right) \rangle_{\text{smp}} \]
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Temporal autocorrelator

Different regimes:

\[ \chi(t, L) \text{ constant} \]

\[ \chi(t, L) \text{ power law, saturation} \]

Crossover between the two regimes:

The transition happens when \( \chi(t_f, L) \) shows dependence on \( L \). This is a lower bound (due to finite size of lattice).
Participation ratio

\[ IPR = \langle \sum_c |\psi_c(t_f)|^4 \rangle_{\text{smp}} \]

for many particles \( IPR \) always decays exponentially with \( L \):
- localized phase \( \rightarrow \) slower than \( V^{-1} \)
- delocalized phase \( \rightarrow \sim V^{-1} \)

Normalized participation ratio \( \eta(t_f, L) = \frac{1}{V IPR(t_f, L)} \propto e^{-\kappa L} \)

Note: the non-interacting AA model has an extended phase which is not fully ergodic: NPR still depends on \( L \).

\( L \): system size; \( V \): size of the configurations space
Entanglement Entropy

Check if a subsystem is a good heat bath. Take Rényi entropy (simpler to compute numerically)

A: sites $0, 1, \ldots, L/2 - 1$; \hspace{1cm} B: sites $L/2, \ldots, L - 1$

$$S_2(t, L) = \langle - \log_2 \left( \text{tr}_A [\rho_A^2(t)] \right) \rangle_{\text{smp}}, \hspace{0.5cm} \rho_A(t) = \text{tr}_B [\rho(t)]$$

Entanglement for

- localized phase: interactions over the boundary AB independent of $L$ in the non-interacting case extensive, but subthermal in the interacting case
- extended phase: $\propto L$, $\rightarrow$ thermal value (finite size: superextensive)
\[ S_2(t_*, L) = mL - S_d \]

\[ S_d = 1.15 \div 1.3 \text{ at high } g \]
Detecting the transition

To get an estimate of the transition point (without finite size scaling)

- autocorrelator: provides a lower bound (due to $L$ finite, $L < \xi$)
- transition point $\rightarrow g$: size dependence begins.
- region dominated by finite size effects has unphysical values (near transition)
  - exponent of normalized participation ratio:
    - transition point $\rightarrow g$: minimal value of $\kappa$
  - entropy:
    - transition point $\rightarrow g$: maximum value of $m$

\[ \eta \propto e^{-\kappa L} \]

\[ S_2 = mL - S_d \]
Phase diagram

Many-body ergodic

Many-body localized

AA localized

AA extended

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Interacting AA model
Conclusion

Interacting AA model:
- peculiar 1D model
- a MBL transition is spotted
- behaviour of some quantities in the MBL phase

Thank you for your attention

References:
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2. Thouless, Niu
   *Wavefunction scaling in a quasiperiodic potential*
3. Tang, Kohmoto
   *Global scaling properties of the spectrum for a quasiperiodic Schroedinger equation*
   Physical Review B 34, 2041 (1986)