Quench Dynamics in Randomly Generated Extended Quantum Models

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Summary
Quantum ergodicity

Von Neumann (1929)

For all initial states, for most of the time and for most *macroscopic* observables

\[ \langle \psi_0(t) | O | \psi_0(t) \rangle \sim \text{Tr}[O \rho_{mc}] = \langle O \rangle_{mc} \]

- No time average needed in the quantum case
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For all initial states, for most of the time and for most *macroscopic* observables

$$\langle \psi_0(t) | O | \psi_0(t) \rangle \simeq \text{Tr}[O \rho_{mc}] = \langle O \rangle_{mc}$$

- No time average needed in the quantum case
- In finite system there are quantum revivals

$$\langle \psi_0(t) | O | \psi_0(t) \rangle = \lim_{T \to \infty} \frac{1}{T} \int_0^T dt \langle \psi_0(t) | O | \psi_0(t) \rangle = \text{Tr}[O \rho_{\text{diag}}]$$

- Diagonal Ensemble $| \psi_0(t) \rangle \langle \psi_0(t) | = \sum_i |c_i|^2 |E_i\rangle \langle E_i| = \rho_{\text{diag}}$
A more modern approach: quantum quench

- Consider a quantum system in an eigenstate $|\psi_0\rangle$ of an Hamiltonian $H$
- At $t = 0$ the Hamiltonian is changed $H \rightarrow H_1$, $|\psi_0\rangle$ evolves according to the new Hamiltonian
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**Fundamental question**

Will the long-time dynamics approach a thermodynamical state?

Technically, only for special initial conditions...

$$\rho_{\text{diag}} = \rho_{\text{mc}} \quad \iff \quad |c_i| = 1/N$$

Single observables have to be considered!
Thermalization for natural observables
Generalities on quantum quench and thermalization

Thermalization for natural observables

Eigenstates Thermalization Hypothesis (Deutsch 92, Srednicki 94)

The expectation value on an eigenstate (EEV) $|E_i\rangle$ of a natural observable $O_i = \langle E_i | O | E_i \rangle$ is a smooth function of its energy $E_i$, being essentially constant on each micro-canonical energy shell.

- Diagonal Ensemble average $O_{Diag} = \sum_i |c_i|^2 O_i$
- $|c_i|^2$ distribution needs to be peaked around $E_0 = \langle \psi_0 | H | \psi_0 \rangle$
Finite-size and rare states

Let’s consider the distribution of the observable in an energy shell with the system size...

Biroli et al. 2010 proved that

\[(\Delta O_e)^2 = \frac{\sum_{i \in e} O_i^2}{N_e} - \left(\frac{\sum_{i \in e} O_i}{N_e}\right)^2 \rightarrow 0 \text{ for } L \rightarrow \infty\]
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- Are the tails of distribution of EEV going to zero? (Rare States)
$\mathbb{Z}_2$-breaking models

\[ H(h) = \begin{pmatrix} A & hB \\ hB^T & C \end{pmatrix} \]

- Strictly even and odd observables

\[
\text{Even} = \frac{1}{L} \begin{pmatrix} A & 0 \\ 0 & C \end{pmatrix}, \quad O = \frac{1}{L} \begin{pmatrix} 0 & B \\ B^T & 0 \end{pmatrix}
\]

- Quench protocol: $-h \rightarrow h$

- We focus on the intermediate case $h = 1$
- \( A, C \rightarrow \mu(M) = \exp \left( \frac{N \text{Tr}(M^2)}{4L^2} \right) \)
- \( B_{ij} \rightarrow \mathcal{N}(0, \frac{2L^2}{N}) \)
- \( L = \ln N \)

\[
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A & hB \\
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\end{pmatrix}
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\[
H(h) = \begin{pmatrix}
A & hB \\
hB^T & C
\end{pmatrix}
\]

$$N(E) = \frac{1}{2\pi L} \sqrt{4L^2 - E^2}$$

$$p(s) = \frac{\pi}{2} se^{-\frac{\pi}{4}s^2}$$
The distribution of the overlaps $|c_i|^2 = |\langle \psi_0 | E_i \rangle|^2$ are broad.

1\textsuperscript{st} prequench state

2000\textsuperscript{th} prequench state
The broad distribution of the overlaps is captured by the

\[ \text{I.P.R.} = \frac{1}{\sum_i c_i^4} \]
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It is the same IPR of a random vector uniformly distributed on the unit \( N \)-sphere:

\[ \frac{1}{\sum_i \langle c_i^4 \rangle} \sim \frac{N}{3} \]
EEVs do not show any energy dependence, their variance and the support of distribution shrinks to zero on the entire spectrum.

\[
\langle e_i | 0 | e_i \rangle 
\]

\[
\text{EEV variance} 
\]

\[
\approx \frac{1}{N} 
\]

\[
\text{EEV max-min} 
\]

\[
\approx \sqrt{\frac{\ln N}{N}} 
\]
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**GOE random matrix**

\[ O_{ii} = \langle E_i \mid O \mid E_i \rangle = \sum_{\alpha} A_{i\alpha} O_{\alpha} \]

\[ O_{\alpha} = O_{\alpha} \mid \alpha \rangle \]

\[ \overline{O_{ii}} = 0 \]

Making the assumption (motivated by the IPR scaling)

\[ A_{i\alpha} \approx \frac{3}{N} \]

and that \( A_{i\alpha} \) are independent

\[ \overline{O_{ii}^2} \approx N \left( \frac{3\sigma O}{N} \right)^2 \propto \frac{1}{N} \]
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For the Max-Min distribution, we use large deviations theory on

\[ \delta_O = 2 \max_i \{ O_{ii} \} \approx \sqrt{\frac{\ln N}{N}} \]
\[ \sigma_{m-d} = \sqrt{(O_{\text{micro}} - O_{\text{diag}})^2} \]

- Thermalization in GOE matrix is always trivially achieved since both the microdiagonal ensemble and the overlaps are randomly sampled among random variables that have a support closer and closer to zero with the system size.
Mimicking a real model

A more physical ensemble

With the GOE ensemble, we obtained trivial thermalization. Can we do better?

- Let’s take the 1D Ising model

\[
H \equiv \sum_{i=1}^{L} \sigma_z^i \sigma_z^{i+1} + h \sigma_x^i = \sum_{i=1}^{L} H_i
\]

- Real space basis:

|↑↑ ... ↑⟩, |↑↑ ... ↓⟩, ... ⇒ |m_1 m_2 ... m_L⟩

- It has the same block structure breaking a $\mathbb{Z}_2$ parity

\[
P = \exp \left[ i\pi \sum_i \left( \sigma_z^i + \frac{1}{2} \right) \right] \quad PH(h)P^\dagger = H(-h)
\]
\[ \langle m_1 \ldots m_N | H_i | m'_1 \ldots m'_N \rangle = (m_i m_{i+1} \delta_{m_i,m'_i} + h \delta_{m_i,-m'_i}) \prod_{k \neq i} \delta_{m_k,m'_k} \]

- \( H \) has \((L + 1)N\) non zero elements
- Let’s count the non-zero elements in the Hamiltonian matrix

\[ \frac{(L + 1)N}{N^2} \propto \frac{\ln N}{N} \ll 1 \]

**Lesson from Ising model**

For a local model, it exists a preferred basis, the real-space basis, where the Hamiltonian matrix and the local observables will be simultaneously sparse.
Idea

Let's generate randomly an adjacency matrix with the proper density of non-zero elements, setting the graph structure of the Hilbert space.
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Sparse random matrix

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Let's generate randomly an adjacency matrix with the proper density of non-zero elements, setting the graph structure of the Hilbert space.

Generate a mask with the proper density of 1

\[ M_{ij} = \begin{pmatrix} 1 & 0 & \ldots & 0 & 1 & \ldots \\ 0 & 1 & \ldots & 0 & 0 & \ldots \\ 0 & 0 & \ldots & 1 & 0 & \ldots \end{pmatrix} \]

- Put random coefficient on the diagonal and off-diagonal links

\[ H_{i < j} = \begin{cases} d_i \text{ with } d_i \text{ drawn from } \mathcal{N}(0, \ln N) & \text{if } i = j \\ o_{ij} \times M_{ij} \text{ with } o_{ij} \text{ drawn from } \mathcal{N}(0, 1) & \text{if } i < j \end{cases} \]

- Change sign after quench in the off-diagonal blocks

\[ E_{gs} = \min_i \{ E_i \} \approx -2 \ln N \]
• Peaked overlap distributions

2000\textsuperscript{th} prequench state

3500\textsuperscript{th} prequench state
EEVs now show some energy dependence

- Even observable
- Odd observable
Consider the IPR of the post-quench eigenbasis w.r.t. the local basis

Reintepret the matrix as a particle hopping on a Bethe-lattice of fixed connectivity $\ln N$ where each site has a random potential (the diagonal part of the matrix) \(\rightarrow\) Anderson model

Mobility edge, even for large system there will be localized states

\[
e_m = \pm \left( 2 - \sqrt{\frac{\pi}{8 \ln \ln N}} \right)
\]
The observables have a structure closely related to $H \rightarrow$ localized post-quench eigenstates are close to eigenstates of the observables $\rightarrow$ EEV $\neq 0$

Delocalized states will overlap with many eigenstates of the observable $\rightarrow$ EEV close to zero

Even observable

Odd observable
- The distribution is shrinking

- But the support of the EEV distribution is not going to zero
Focusing on an energy window around $e = 0$

- Rare states seem to exist also in the center of the spectrum ($c \neq 0$ in $a/x^b + c$ fit)
Focusing on initial states with energy close to $e = 0$

$$\sigma_{m-d}^2 = (O_{\text{micro}} - O_{\text{diag}})^2$$

- Thermalization occurs also for sparse matrices, but only because the quench protocol seems to provide initial states with no large overlap on far-from-average states.
\[ \mathcal{O}_t = \langle \psi_0(t) | \mathcal{O} | \psi_0(t) \rangle = \text{Tr}(\mathcal{O} \rho_{mc}) \quad \text{for almost all } t, \mathcal{O}. \]
\[ \mathcal{O}_t = \langle \psi_0(t) | \mathcal{O} | \psi_0(t) \rangle = \text{Tr}(\mathcal{O} \rho_{mc}) \quad \text{for almost all } t, \mathcal{O}. \]

To reach the asymptotic value it needs at least:

\[ \tau_s = \min\{ t \mid \mathcal{O}_t = \mathcal{O}_\infty = \text{Tr}(\mathcal{O} \rho_{\text{diag}}) \}. \]

But if we want to argue \'a la\ Von Neumann, we need to look at fluctuations:

\[ \Delta^\mathcal{O}_t \equiv \frac{1}{t} \int_0^t (\mathcal{O}_t - \mathcal{O}_\infty)^2 d\tau \approx e^{-t/\tau_F} \]

\[ \frac{\mu(\tau \in [0, t] \text{ and } |\mathcal{O}_\tau - \mathcal{O}_\infty| > a)}{t} < \frac{\Delta^\mathcal{O}_t}{a^2}. \]
Fluctuations are much slower, living on a time scale related to the minimum gap between energies

$$\tau_F \to \frac{1}{\min_{i \neq j} (E_i - E_j)} \propto N$$

While, the value of the observable reached its asymptotic average much before, with a scaling related to the system volume

$$\tau_S \propto \ln N \simeq L$$
ETH seems to be the mechanism behind thermalization in random models

Locality, in the guise of sparseness of the matrix representation, plays a major towards the creation of physically realistic random models

It is possible to identify two different thermalization time scales, a short one, \( \tau_S \), and a long one, \( \tau_F \), with different dependence upon the system size.
Thank you!