An introduction to tensor-network states and MERA

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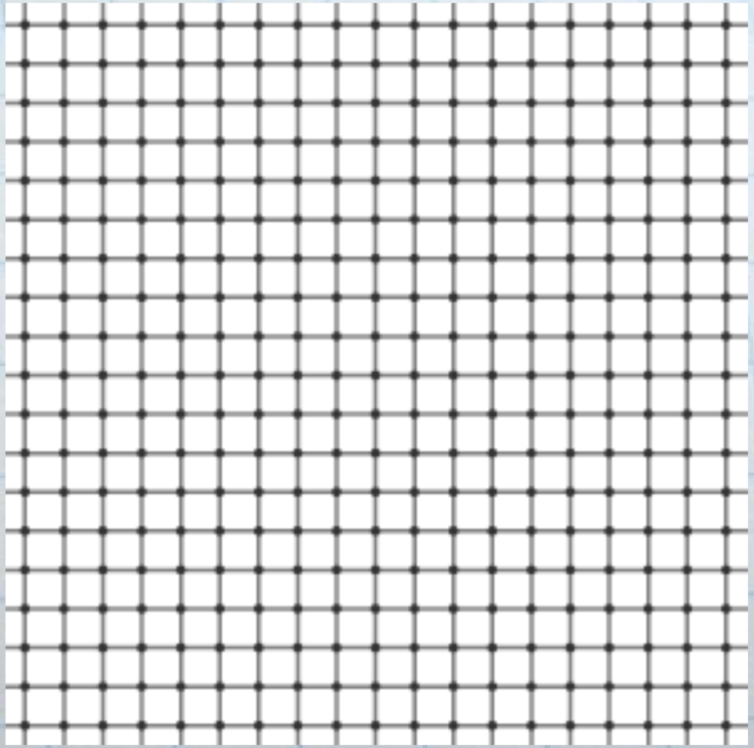
A typical problem

We are given:

- A lattice with **N** sites
- On each site a $\mathbf{C}^{\mathbf{d}}$ hilbert space
- A quantum hamiltonian

The most general state:

$$|\Psi\rangle = \sum_{i_1, \ldots, i_n} \mathcal{T}_{i_1 \ldots i_n} |i_1 \ldots i_n\rangle$$



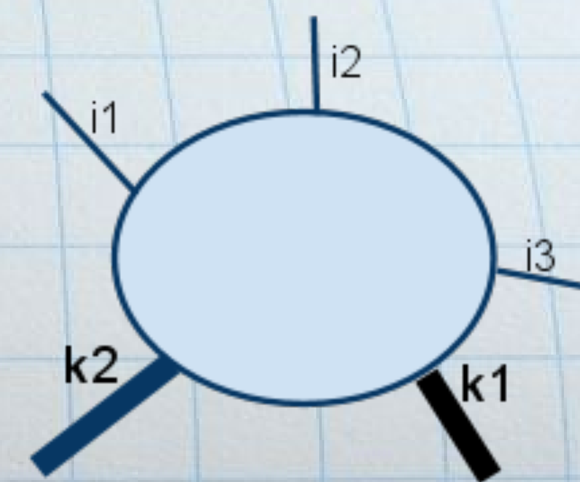
Exponentially large number of states: $\mathbf{d}^{\mathbf{N}}$

Can we do better?

Graphical representation

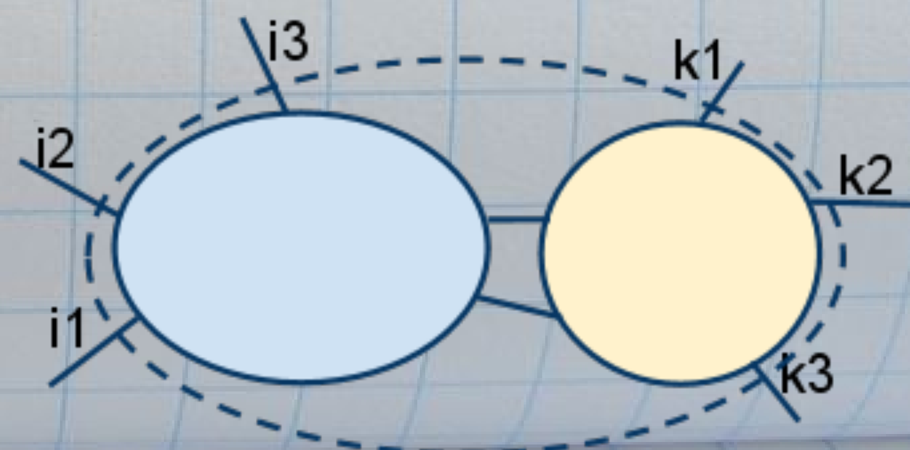
- We have to deal with an **N** index tensor!
- Is there a good way to represent it?

$$\mathcal{T}_{i_1 i_2 i_3 k_1 k_2 k_3}$$



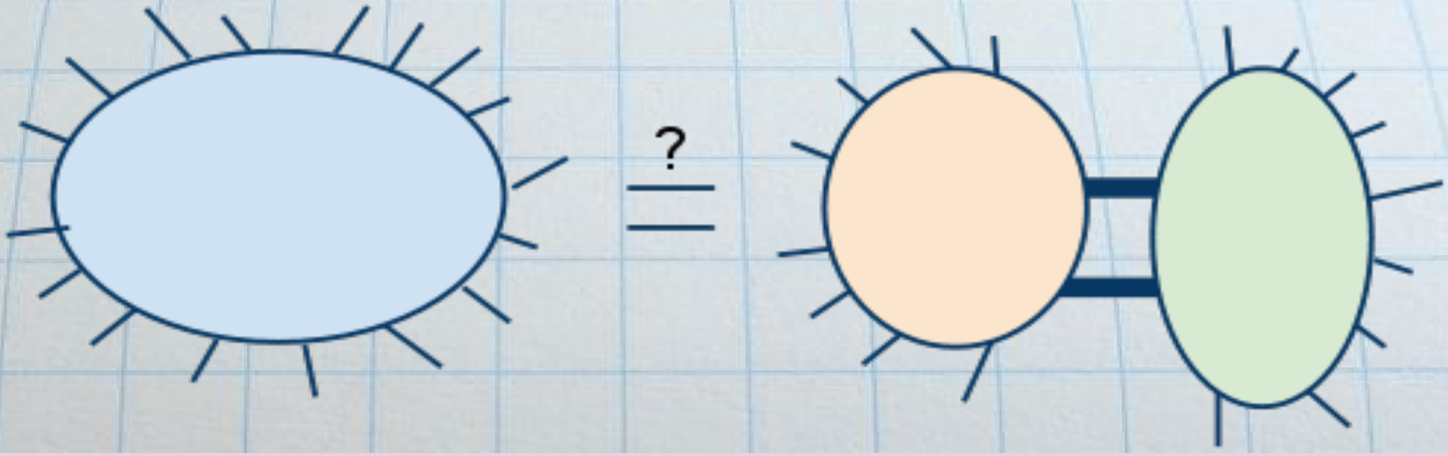
Also the product can be implemented:

$$\sum_{\alpha,\beta} \mathcal{T}_{i_1 i_2 i_3 \alpha \beta} \mathcal{G}_{k_1 k_2 k_3 \alpha \beta}$$



How this can help us?

IDEA: We can try to "factorize" the general tensor!



If we can suppose the physically relevant states are factorized we can reduce the computational cost!

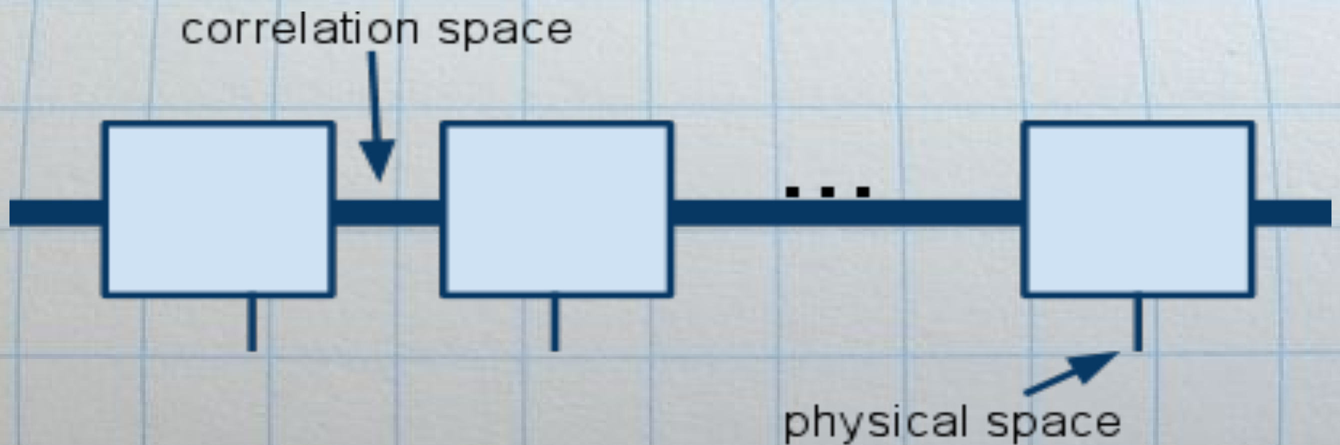
What can drive us in this graphical choice?

SYMMETRIES!

Matrix product states

Let's specialize our problem to

- **1D spin system: local space C^2**
- **Traslation invariant model**



- The correlation space is chosen as $\mathbf{C}^p$
- Each square is $\mathbf{A}_{aij,\ a = 1,2;\ i,j = 1,\dots,p}$
- for $\mathbf{p = logN}$ it can describe all the possible states

How it works practically?

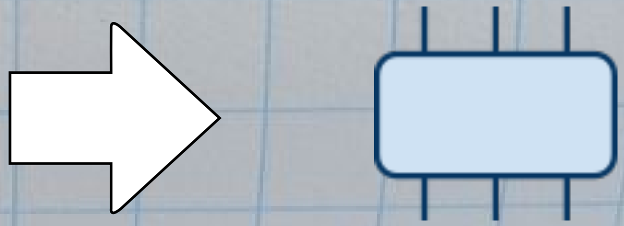
- We write the state as function of $\mathbf{A}_{aij}$
- One usually works for fixed finite **p**: error is logarithmic
- Through a variational approach we determine the best **A**

$$E_0 = \min_{A} \langle \Psi(A) | H | \Psi(A) \rangle$$

How to compute correlation functions?

Also operator will have their graphical representation:

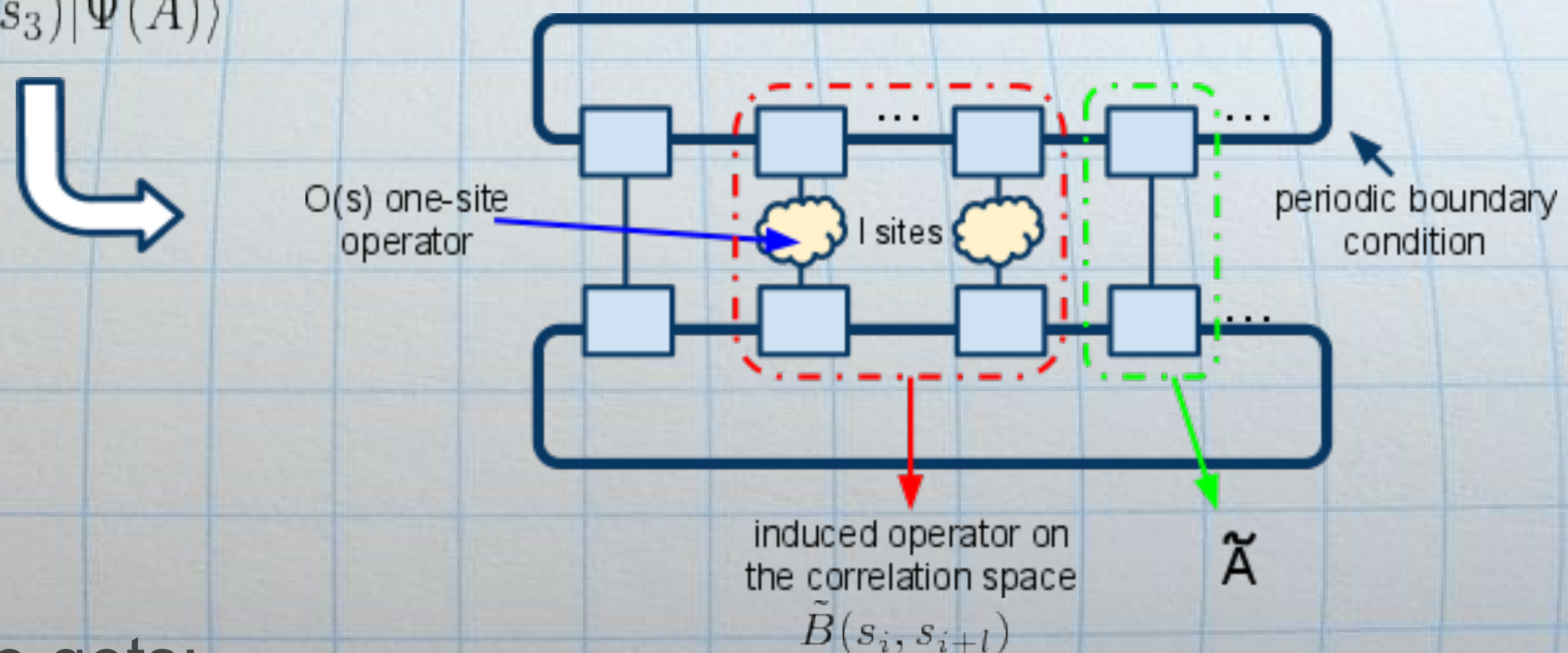
$$B(s_1, s_2, s_3)$$



MPS are off-critical

For fixed **p** they cannot describe the ground state of a critical Hamiltonian.

$\langle \Psi(A) | B(s_1, s_2, s_3) | \Psi(A) \rangle$



In the TL one gets:

$$\mathrm{Tr}(\widetilde{A}^{\infty} \widetilde{B}\, \widetilde{A}^{l} \widetilde{B}') = \sum_{\mathrm{Sp}(A)} \lambda^{l} \mathrm{Poly}_{BB'A}(l)$$

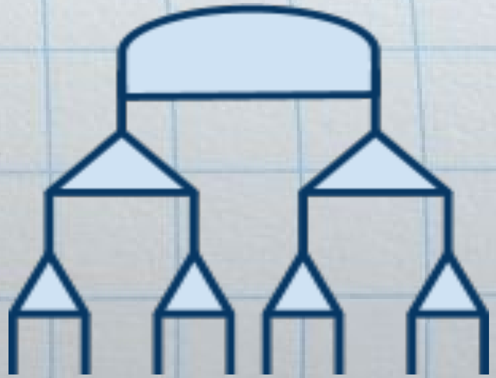
What to do to have criticality?

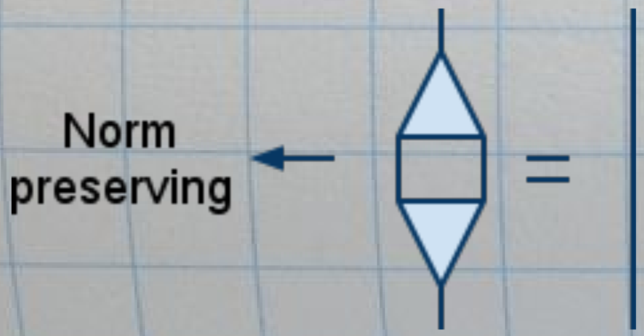
The symmetry is the scale invariance!

If we want it to be implemented at the tensor level:

AUTOSIMILARITY

Simplest attempt:
TENSOR TREE NETWORK





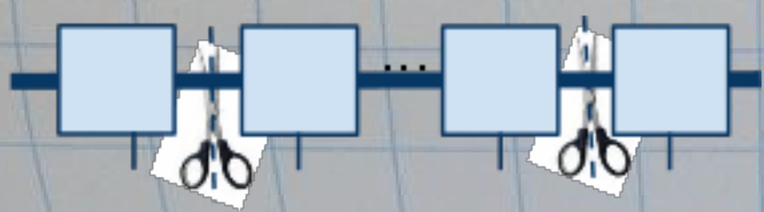
How good is TTN?

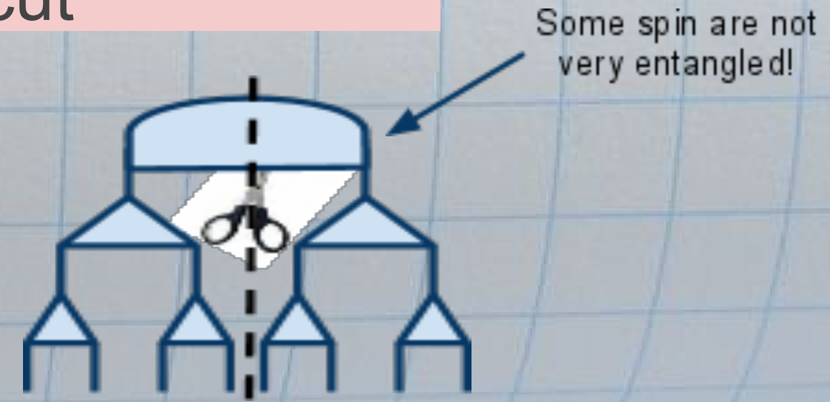
Let's look entanglement entropy:

Area Law:

$S(l) \sim S_{max}$	1d non critical
$S(l) \sim c \log l / 6$	1d critical
$S(l) \sim K\, l^{D-1}$	D-dimensional system

In tensor network, for fixed size of inner space, entanglement entropy can be estimated as the number of bonds to be cut





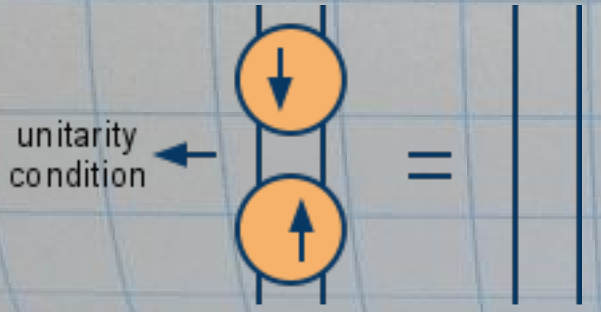
MERA States

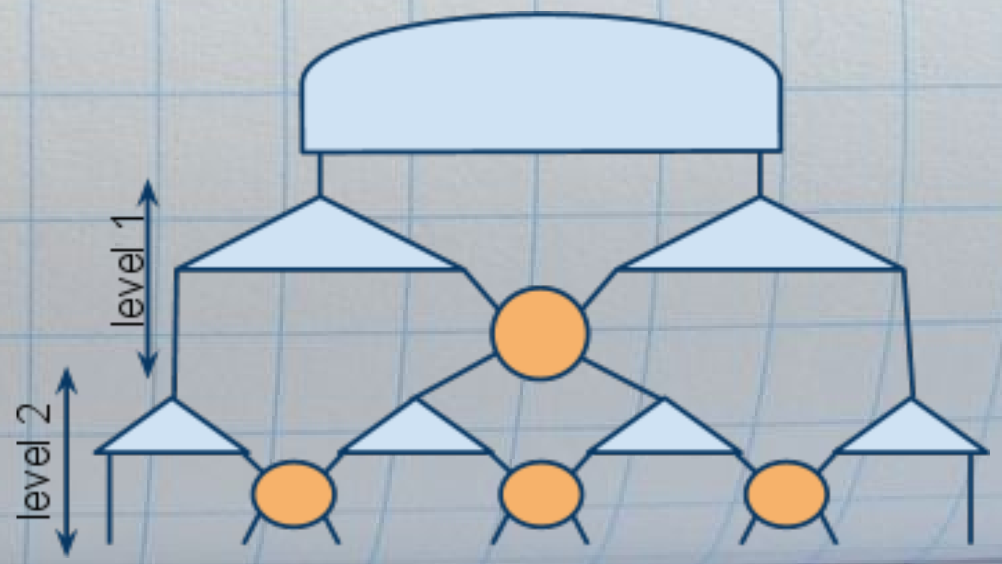
TTN cannot work because some spins are treated in a different way:

- **H_1** and **H_2** describing the same phase, but differing **short distances** may give different results!
- Non-critical system can be well described: Entanglement entropy saturates!
- Also for critical error is not so large (logarithmic)

HOW CAN WE DO BETTER?

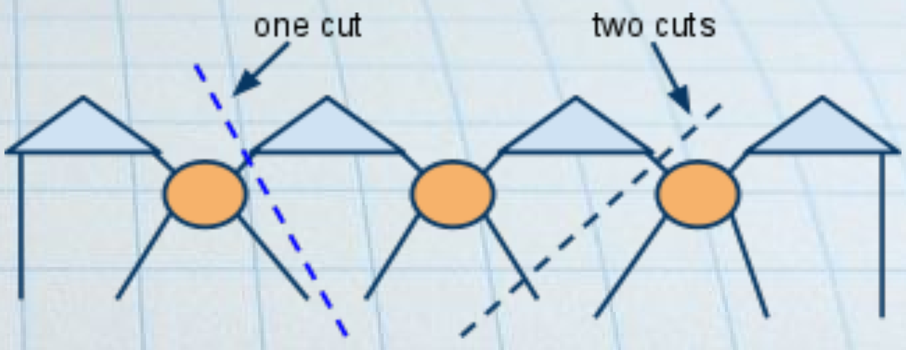
We add disentanglers!





How goes entanglement?

As before we count cuts:



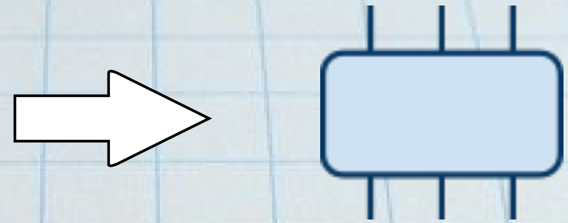
Roughly speaking **3 cuts per level**!

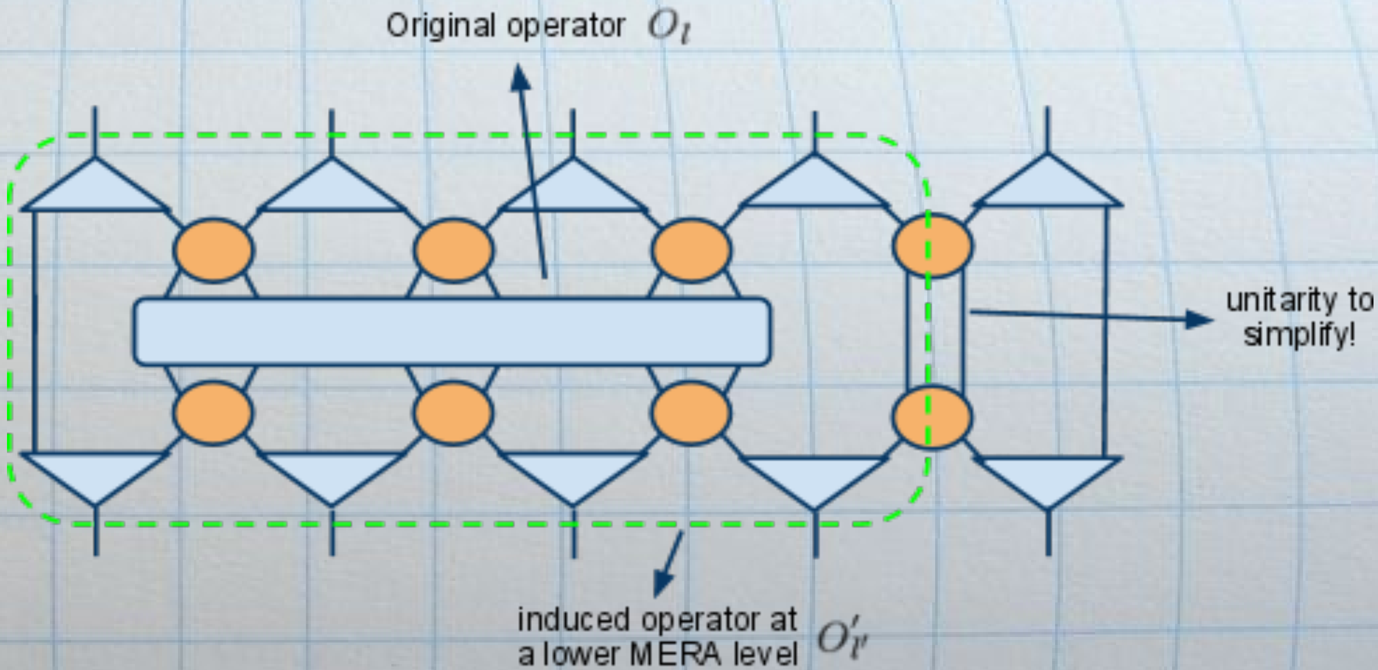
$$S \simeq 3 \log N \cdot S_3$$

Now it has the correct **1d** critical behavior!

Is it possible to work with observables?

O_l





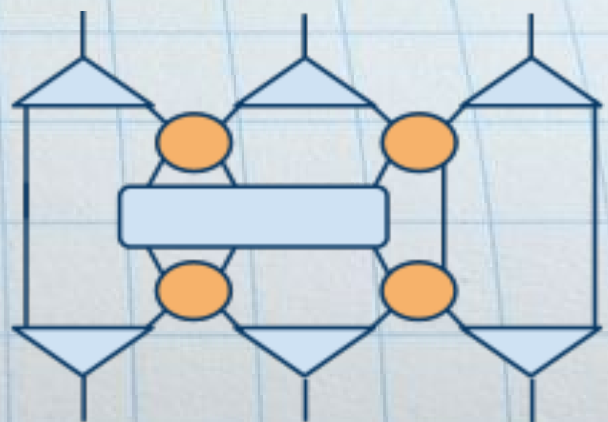
To compute mean we have to sandwich:

$$\langle \Psi^{(n)}_{MERA} | O_l | \Psi^{(n)}_{MERA} \rangle = \langle \Psi^{n-1}_{MERA} | O'_{l'} | \Psi^{(n-1)}_{MERA} \rangle$$

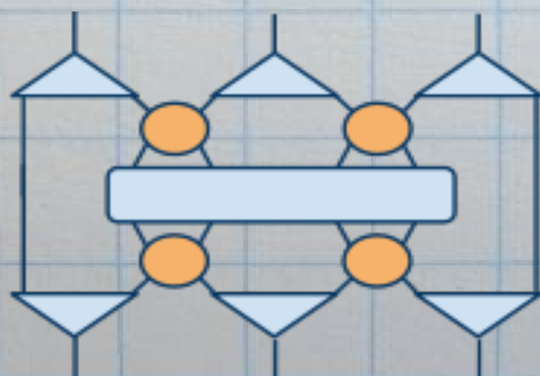
Superoperator and stable structures

- It comes out a map between operators: $O_l = \mathcal{A}[O'_{l'}]$
- $l' \leq l$: operators always become more local
- Some stable structure exists:

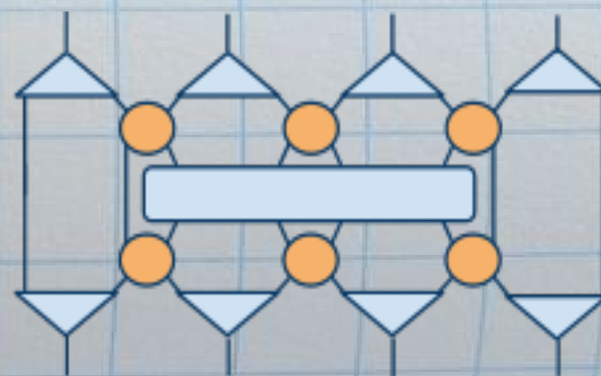
3 sites always goes in 3 sites!



4 sites are metastable:



$4 \to 3$



$4 \to 4$

General results: $2l+1 \to l+2$ $\quad$ $2l \to \begin{matrix} l+2 \\ l+1 \end{matrix}$

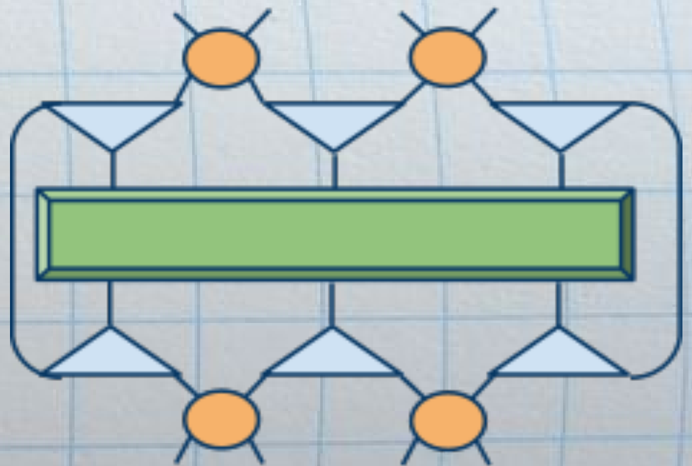
Density matrix superoperator

As usual from operators one can pass to states:

$$\langle O_3\rangle = \mathrm{Tr}[\rho^{(n)}_{3,j} O_3] = \mathrm{Tr}[\rho^{(n-1)}_{3,j} \mathcal{A}_j(O_3)] = \mathrm{Tr}[\mathcal{D}_j(\rho^{(n-1)}_{3,j}) O_3]$$

$\mathrm{Tr}(A^\dagger B)$ is a scalar product

The dual of the ascending operator is the **descending** one acting on density matrices



$\mathcal{D}_{3\to 4}(\rho^{(n)}_3) = \rho^{(n+1)}_4$

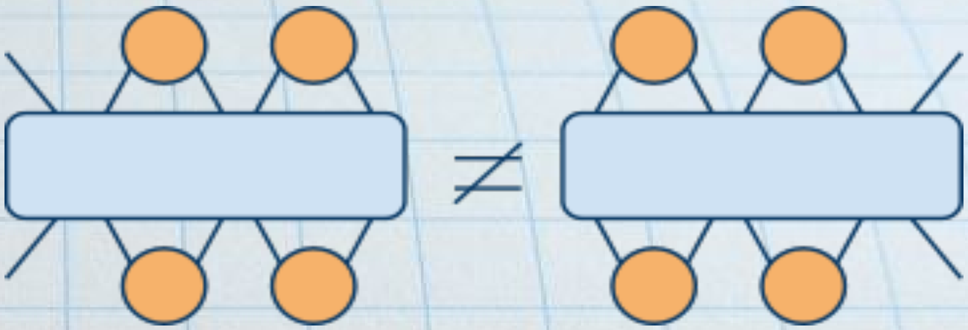
Properties of D:

- completely positive
- trace preserving

It's the most general quantum evolution

Using traslation symmetry

Take care: site position counts!



$$\mathcal{D}_L, \mathcal{D}_R, \mathcal{A}_L, \mathcal{A}_R$$

A lot of different objects: how to take care of this indexes?

use traslation symmetry

$$\langle \overline{O_3} \rangle = \frac{1}{N} \sum_{\nu}^{N} \langle O_{\nu,\nu+1,\nu+2} \rangle = \mathrm{Tr}(O \overline{\rho_3}^{(n)})$$

$$\overline{\rho_3^{(n)}} = \frac{1}{N} \sum_{\nu}^{N} \rho_{\nu,\nu+1,\nu+2}^{(n)}$$

$$\overline{\rho_3^{(n)}} = \overline{\mathcal{D}}(\overline{\rho}_3^{(n-1)})$$

$$\overline{\mathcal{D}} = \frac{1}{2}(\mathcal{D}_L + \mathcal{D}_R)$$

Thermodynamic limit and RG flow

Now the descending superoperator is always the same!

FIXED POINT → **SCALE INVARIANT POINT!**

$$\rho_3^F = \overline{D}\,\rho_3^F$$

You can access numerically at the thermodynamic limit

Critical exponent

We can see that now it works well for critical points:

$$C^{(n)}_{\Delta\alpha}(O,O') = \frac{1}{N}\sum_{\nu} \langle O_{\nu} O'_{\nu+\Delta\alpha}\rangle = \mathrm{Tr}[O\otimes O'\,\overline{\eta^{(n)}_{\Delta\alpha}}\,]$$

3 plus 3 sites mean density matrix

Computing graphically one gets:

$$\overline{\eta_{\Delta\alpha}}^{(n)} = \overline{\mathcal{D}}(\overline{\eta^{(n-1)}_{\Delta\alpha/2}})$$

Points get closer!

$$= \mathrm{Tr}[O\otimes O'\,\overline{\mathcal{D}}^{s}(\eta_3^{(n-s)})] \underset{TL}{=} \mathrm{Tr}[O\otimes O'\,\overline{\mathcal{D}}^{s}(\rho_6^F)]$$

$$= \sum_{\lambda\in \mathrm{Sp}(\overline{\mathcal{D}})} (\Delta\alpha)^{\log\lambda}\,\mathrm{Poly}(\log \Delta\alpha)$$

critical exponents

CFT and primary fields

What happens if you look $\overline{\mathcal{A}}$ eigenvectors?

$$\mathrm{Tr}[O\otimes O'\overline{\mathcal{D}}^{s}\,\overline{\eta_6^F}] \underset{adj}{=} \mathrm{Tr}[\overline{\mathcal{A}}^{s}\,O\otimes O'\,\overline{\eta_6^F}]$$

$$=\lambda^{s}\mathrm{Tr}[O\otimes O'\,\overline{\eta_6^F}]$$

you can get anomalous dimensions

$\overline{\mathcal{A}}$ → eigenvectors are primary fields, eigenvalues are anomalous dimensions

$\overline{\mathcal{D}}$ → modulus 1 eigenvector is the scale invariant critical state, eigenvalues gives critical exponents

Some references...

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- "Entanglement renormalization, scale invariance, and quantum criticality" Robert N. C. Pfeifer, Glen Evenbly, Guifré Vidal, 0810.0580
- "Critical exponents of one-dimensional quantum critical models by means of MERA tensor network", R. Fazio, S. Montangero, M. Rizzi, V.Giovannetti, 0810.1414
- "Critical properties of homogeneous binary trees", P. Silvi, V.Giovannetti, S.Montangero, M.Rizzi, J. I. Cirac, R. Fazio

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