Thermally Increasing Correlation/Modulation Lengths and Other Selection Rules in Systems with Long Range Interactions

March 27, 2009
1. Modulated patterns in physics

2. Phenomenological approach

3. Large-$n$ approach

4. The selection rule

5. Some remarks
Modulated patterns in physics

- superconductors ($\approx 10\mu m$)
- phospholipides ($\approx 25\text{nm}$)
- chemical reactions ($\approx .3\text{mm}$)
- convection ($\approx 1\text{cm}$)
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Modulated patterns in physics

Stripes, fingers, bubbles and the like
Phenomenological approach

Ginzburg-Landau functional

$$\mathcal{F}_\phi = \mathcal{F}_0[\phi] + \frac{b}{2} \int d^2 r |\nabla \phi(r)|^2$$

- "mexican hat" component
- surface tension component
Phenomenological approach

Ginzburg-Landau functional

$$\mathcal{F}_\phi = \mathcal{F}_0[\phi] + \frac{b}{2} \int d^2r |\nabla \phi(r)|^2$$

"mexican hat"

surface tension

$$- \frac{Q}{2} \int \int d^2r \, d^2r' \phi(r)g(r - r')\phi(r')$$

long-range int.
Phenomenological approach

Ginzburg-Landau functional

\[ \mathcal{F}_\phi = \mathcal{F}_0[\phi] + \frac{b}{2} \int d^2 r |\nabla \phi(r)|^2 \]

- surface tension

\[ - \frac{Q}{2} \int \int d^2 r \ d^2 r' \phi(r)g(r-r')\phi(r') \]

- long-range int.

\[ + \frac{\kappa}{2} \int d^2 r (\nabla^2 \phi(r))^2 \]

- curvature effects
Ginzburg-Landau functional

\[ \mathcal{F}_\phi = \mathcal{F}_0[\phi] + \frac{b}{2} \int d^2 r |\nabla \phi(r)|^2 \]

"mexican hat" surface tension

\[ - \frac{Q}{2} \int \int d^2 r d^2 r' \phi(r) g(r - r') \phi(r') \]

long-range int.

\[ + \frac{\kappa}{2} \int d^2 r (\nabla^2 \phi(r))^2 + \ldots \]

curvature effects
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Phenomenological approach

One more picture

Magnetic garnet film as $T \uparrow$ period increases

Langmuir film of DMPA and cholesterol as $T \downarrow$ period decreases
Large-\(n\) model

\[
H = \frac{1}{2} \sum_{\vec{x}, \vec{y}} S(\vec{x}) V(\vec{x}, \vec{y}) S(\vec{y})
\]

Spins satisfy the mean spherical constraint

\[
\sum_{\vec{x}} \langle S(\vec{x})^2 \rangle = N
\]
Large-\textit{n} model

\[ H = \frac{1}{2} \sum_{\vec{x}, \vec{y}} S(\vec{x}) V(\vec{x}, \vec{y}) S(\vec{y}) \]

Spins satisfy the mean spherical constraint \( \sum_{\vec{x}} \langle S(\vec{x})^2 \rangle = N \)

Introduce a Lagrange multiplier \( \mu \) to enforce the constraint

- Fourier transform of the interaction kernel \( v(\vec{k}) \)

\[ 1 = k_B T \int \frac{d^d k}{(2\pi)^d} \frac{1}{\mu + v(\vec{k})} \]

- Critical temperature \( T_c \) (if any!)

\[ (k_B T_c)^{-1} = \int \frac{d^d k}{(2\pi)^d} \frac{1}{\mu_c + v(\vec{k})} \]

where \( \mu_c = - \min_{q \in BZ} v(q) \).
Correlation functions

\[ G(\vec{x}) \equiv \langle S(0)S(\vec{x}) \rangle = k_B T \int \frac{d^d k}{(2\pi)^d} \frac{e^{i\vec{k} \cdot \vec{x}}}{v(\vec{k}) + \mu} \]

- Normalization of \( G(\vec{x} = 0) = 1 \) fixes \( \mu \).
- Rotationally invariant system: \( v(\vec{k}) \) is a function of \( k^2 \)
Correlation functions

\[ G(\vec{x}) \equiv \langle S(0)S(\vec{x}) \rangle = k_B T \int \frac{d^d k}{(2\pi)^d} \frac{e^{i \vec{k} \cdot \vec{x}}}{\nu(\vec{k}) + \mu} \]

- Normalization of \( G(\vec{x} = 0) = 1 \) fixes \( \mu \).
- Rotationally invariant system: \( \nu(\vec{k}) \) is a function of \( k^2 \)

Correlation functions calculation \( \Leftrightarrow \) finding poles of the integrand!
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Large-$n$ approach

Correlation functions

\[ G(\vec{x}) \equiv \langle S(0)S(\vec{x}) \rangle = k_B T \int \frac{d^d k}{(2\pi)^d} \frac{e^{i\vec{k} \cdot \vec{x}}}{v(\vec{k}) + \mu} \]

- Normalization of \( G(\vec{x} = 0) = 1 \) fixes \( \mu \).
- Rotationally invariant system: \( v(\vec{k}) \) is a function of \( k^2 \)

Correlation functions calculation \( \iff \) finding poles of the integrand!

Assume \( k^s[v(k) + \mu] \) is a polynomial

\[ P(z) = \sum_{m=0}^{M} a_m z^m \]

in \( z = k^2 \iff \) correlator displays a net of \( M \) correlation and modulation lengths.

- Different length scales arise for any \( M \geq 2 \)
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Large-$n$ approach

Short and long range interactions

Long range
Short and long range interactions

Long range screened LR interaction may be emulated by the FT kernel $[k^2 + \lambda^2]^{-p}$. E.g. Coulomb screened

$$d = 3, \hspace{0.5cm} V = \frac{1}{8\pi} \frac{1}{|\vec{x} - \vec{y}|} e^{-\lambda|\vec{x} - \vec{y}|}$$

$$d = 2, \hspace{0.5cm} V = \frac{1}{4\pi} \ln |\vec{x} - \vec{y}| e^{-\lambda|\vec{x} - \vec{y}|}$$

$$\implies v(k) = [k^2 + \lambda^2]^{-1}$$
Short and long range interactions

**Long range**

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**Short range**
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Large-$n$ approach

Short and long range interactions

**Long range**

Screened LR interaction may be emulated by the FT kernel $[k^2 + \lambda^2]^{-p}$. E.g. Coulomb screened

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\]

\[
d = 2, \quad V = \frac{1}{4\pi} \ln|\vec{x} - \vec{y}| e^{-\lambda|\vec{x} - \vec{y}|}
\]

\[\Rightarrow v(\vec{k}) = [k^2 + \lambda^2]^{-1}\]

**Short range**

Lattice Laplacian: $\Delta(\vec{k}) = 2 \sum_{l=1}^{d}(1 - \cos k_l)$

In real space:

\[
\langle \vec{x}|\Delta|\vec{y}\rangle = \begin{cases} 
2d & \text{for } \vec{x} = \vec{y} \\
-1 & \text{for } |\vec{x} - \vec{y}| = 1 
\end{cases}
\]

In the continuum limit $\Delta \rightarrow k^2$
LR-SR competition: an example

- short-range attractive interaction
- long-range screened Coulomb interaction

Fourier transform of the interaction kernel

\[ \nu(k) = k^2 + \frac{Q}{k^2 + \lambda^2} \]

(screened model of frustrated phase separation in cuprates). Pole dynamics controls evolution of correlation lengths.
Correlation function ($d = 3$)

At high temperatures ($T > T^*$ where $\mu(T^*) = \lambda^2 + 2\sqrt{Q}$)

$$G(\vec{x}) = \frac{k_B T}{4\pi|\vec{x}|} \frac{1}{\beta^2 - \alpha^2} [(\lambda^2 - \alpha^2)e^{-\alpha|\vec{x}|} - (\lambda^2 - \beta^2)e^{-\beta|\vec{x}|}]$$

For $T < T^*$

$$G(\vec{x}) = \frac{k_B T}{8\alpha_1\alpha_2\pi|\vec{x}|} e^{-\alpha_1|\vec{x}|}$$

$$[((\lambda^2 - \alpha_1^2 + \alpha_2^2)\sin \alpha_2|\vec{x}| + 2\alpha_1\alpha_2 \cos \alpha_2|\vec{x}|]$$

where

$$\alpha^2, \beta^2 = \frac{\lambda^2 + \mu \pm \sqrt{(\mu - \lambda^2)^2 - 4Q}}{2}$$

and $\alpha = \alpha_1 + i\alpha_2 = \beta^*$

(Poles located at $k = i\alpha, i\beta$)
SR-SR competition

FT of the interaction kernel

\[ \nu(k) = a_4 k^4 - a_2 k^2 \]

Teubner-Stray correlator

\[ G^{-1}(\vec{k}) = a_2 k^4 - a_1 k^2 + \mu \]

\[ G(\vec{x}) \approx \frac{\sin \kappa |\vec{x}|}{\kappa |\vec{x}|} e^{-|\vec{x}|/\xi} \]

where

\[ \kappa = \sqrt{\sqrt{\mu/4a_2} + a_1/4a_2} \]

\[ \xi = \sqrt{\sqrt{\mu/4a_2} - a_1/4a_2} \]
The selection rule

\[ G(x) = \frac{1}{x^{d-2}} \left( A_1 e^{-x/\xi_1} + A_2 e^{-x/\xi_2} + \ldots \right) \]

where at least one \( \xi_i \) diverges.

At low temperatures, \( G(x) \approx \frac{\cos(\kappa x)}{x^{d-2}} e^{-x/\xi} + \ldots \) i.e. correlations turn into modulation lengths.

Modulation length *increase* as \( T \) is raised.
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The selection rule

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G(x) = \frac{1}{x^{d-2}} \left( A_1 e^{-x/\xi_1} + A_2 e^{-x/\xi_2} + \ldots \right)
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modulation length \textbf{increase} as \( T \) is raised

\textbf{LR-SR}

- High-\( T \) limit \( G(x) = \frac{1}{x^{d-2}} \left( A_1 e^{-x/\xi_1} + A_2 e^{-x/\xi_2} + \ldots \right) \)
- At low temperatures \( G(x) \approx \cos(\kappa x) x^{d-2} e^{-x/\xi} + \ldots \) i.e. correlations turn into modulation lengths

\textbf{SR-SR}

- modulation and correlation maintain their identity as \( T \) is varied
- modulation length \textbf{decrease} as \( T \) is raised

... summarizing
Thermally Increasing Correlation/Modulation Lengths and Other Selection Rules in Systems with Long Range Interactions

The selection rule

...summarizing

LR-SR

High-$T$ limit $G(x) = \frac{1}{x^{d-2}}(A_1 e^{-x/\xi_1} + A_2 e^{-x/\xi_2} + \ldots)$ where at least one $\xi_i$ diverges

At low temperatures $G(x) \approx \cos(\kappa x) e^{-x/\xi} + \ldots$ i.e. correlations turn into modulation lengths

modulation length increase as $T$ is raised

SR-SR

modulation and correlation maintain their identity as $T$ is varied

modulation length decrease as $T$ is raised

SELECTION RULE
Some remarks

- $1/n$ corrections do not alter substantially this picture
- Lattice effects raise $T_c$ from 0 to a finite value
References