Unusual corrections to scaling in entanglement entropy

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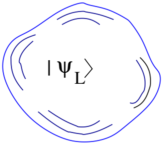
Summary

- Entanglement entropies in 1+1 QFT (review)
- Usual and unusual corrections to scaling in Entanglement entropies

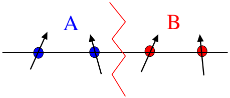
Entanglement (again)



Hall fluid,...

- Consider a many body state $|\psi\rangle$, i.e. the g.s. of a many body Hamiltonian H.

- My simplest example is the g.s. of a spin chain
  Simplest because: $\mathcal{H} = \otimes_{i=1}^{N} \mathcal{H}_i$,



$$|\psi\rangle = \sum_{i_1,\ldots i_N=\pm} c_{i_1,\ldots,i_N} |i_1,\ldots,i_N\rangle = \sum_{j=1}^{d_B}\sum_{i=1}^{d_A} c_{ij} |\psi_i^A\rangle |\phi_j^B\rangle$$

- We allow manipulations on $\mathcal{H}_A$ ($U(d_A)$) and $\mathcal{H}_B$, ($U(d_B)$) and ask:
  How much $|\psi\rangle$ is factorized?
  Useful quantity:

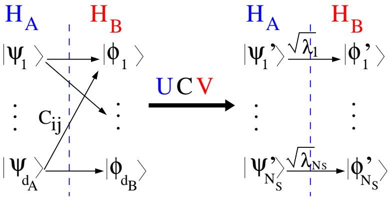
$$(\rho_A)_{ij} = (\mathrm{Tr}_B |\psi\rangle\langle\psi|)_{ij} = \sum_{k=1}^{d_B} c_{ik} c^*_{kj} = \left(cc^\dagger\right)_{ij}$$

Entanglement entropies (again)

- After partitioning the Hilbert space $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$

$$|\psi\rangle = \sum_{i=1}^{d_A} \sum_{j=1}^{d_B} c_{ij} |\psi_i^A\rangle |\phi_j^B\rangle$$

- Diagonalizing the matrix C:



- Trivial fact: λ_i are eigenvalues of ρ_A (ρ_B)

$$|\psi\rangle = \sum_{i=1}^{N_s} \sqrt{\lambda_i} |\psi_i'\rangle |\phi_i'\rangle$$ Schmidt decomposition

- Quantify how much λ_i are spread inside $|\psi\rangle$:

$$S_n(|\psi\rangle) = \frac{1}{1-n} \log \mathrm{Tr} \rho_A^n$$

A question...

Question:

- Q: Given c_{ij}, randomly distributed on a sphere $d_A \times d_B$ how it is distributed the Von Neumann entropy $S_1\left(|\psi\rangle\right)$?

Answer: [Lewenstein, Les Houches lectures 2009]

- If $d_A \ll d_B$:

$$\langle S_1\left(|\psi\rangle\right)\rangle \sim \log d_A - \frac{d_A}{2d_B} \qquad \text{(volume law)},$$

- Means: A many body random state is typically maximally entangled (of course physical states are eigenstates of Hamiltonian...)

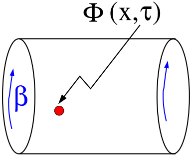
Replicas and Reny entropies

- Consider a $d = 1 + 1$ QFT at finite temperature $\frac{1}{\beta}$

$$\rho = \frac{1}{Z} \sum_{\text{states}} e^{-\beta E_n} |E_n\rangle\langle E_n| \xrightarrow{\beta \to \infty} |0\rangle\langle 0|$$

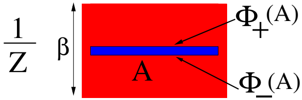
- Partition function can be represented as:

$$Z = \int \mathcal{D}\Phi e^{-\int_0^\beta d\tau \mathcal{L}_E[\phi]}$$



- The reduced density matrix is:

$$\rho_A\left(\phi_+(x, 0^+); \phi_-(x, 0^-)\right)$$



- In the limit $\beta \to \infty$:

$$\mathrm{Tr}\rho_A^n = \frac{Z_n}{Z^n},$$ Partition function on a n-sheeted Riemann surface

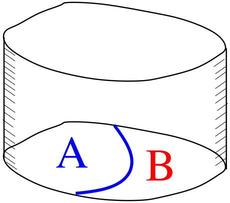
Scaling of the Reny entropies

- In particular Reny entropies are free energies on Riemann surfaces:

$$S_n \propto \log \mathrm{Tr}\rho_A^n = -[F_n - nF_1]$$

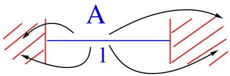
- Rk: Terms $o\left(\varepsilon^{-2}\right)$ and $o\left(\varepsilon^{-1}\right)$ cancel in S_n (volume and area). Indeed:

$$S_n \sim A\varepsilon^{-d+2}, \qquad \text{(Area law)}$$



- For a $1+1$ CFT:

$$S_n = -\frac{1}{1-n}(F_n - nF_1) = \frac{c}{6}\left(1+\frac{1}{n}\right)\log\frac{l}{\varepsilon}$$



- Huge correlations due to massless particles (how? which?)

The proof for a 1+1 CFT

- There are (at least) three equivalent ways to get the 1+1 CFT result:

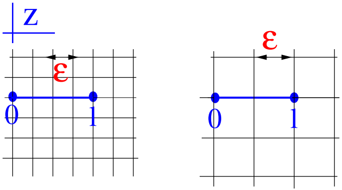
$$S_n = \frac{c}{6}\left(1+\frac{1}{n}\right)\log\frac{l}{\varepsilon}$$

I chose this [Cardy, Les Houches lectures 2008]:

1. Apply a dilatation to the lattice:

$$\varepsilon \to (1+\lambda)\varepsilon$$

$$\varepsilon\frac{\partial F_n}{\partial \varepsilon} = -\frac{1}{2\pi}\int_{\mathcal{R}_n} d^2z \langle \Theta(z,\bar{z})\rangle_{\mathcal{R}_n}$$

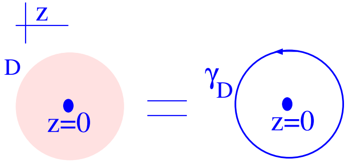


2. However $\langle \Theta(z,\bar{z})\rangle_{\mathcal{R}_n} = 0$, but at the branch points where:

$$\langle T(z)\rangle_{\mathcal{R}_n} \sim \frac{1}{z^2}\frac{c}{24}\left(1-\frac{1}{n^2}\right).$$

Result follows from:

$$\frac{1}{2\pi}\int_D d^2z \langle\Theta\rangle_{\mathcal{R}_n} = -i\frac{n}{2\pi}\oint_{\gamma_D} dz z\langle T(z)\rangle_{\mathcal{R}_n} + \text{c.c.}$$



Corrections to the (Field Theory) scaling limit for Reny Entropies

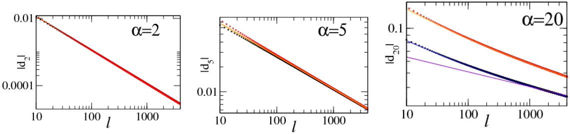
- The CFT result holds in the scaling limit $l \gg \varepsilon$. However lattice effects can produce corrections:

$$S_n(l) = \frac{c}{6}\left(1 + \frac{1}{n}\right)\log\frac{l}{\varepsilon} + \sum_{\alpha<0} A_\alpha \left(\frac{l}{\varepsilon}\right)^{\alpha}$$

- In [Calabrese et al. 2010] an analytic calculation of subleading term in $\frac{l}{\varepsilon}$ is reported for XX spin chain in zero magnetic field ($c = 1$).

$$S_n(l) = S_n^{CFT}(l) + (-1)^l f_n \left(\frac{l}{\varepsilon}\right)^{-\frac{2}{n}}$$

- That was checked numerically ($\alpha = n$):

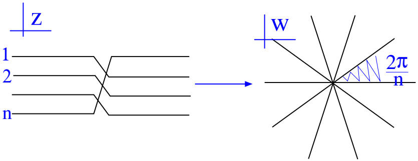


- Can Field Theory predict this power?
- Rk: Actually XX Reny entropies are just twice as Ising!

Detour: Two points function on a Riemann surface

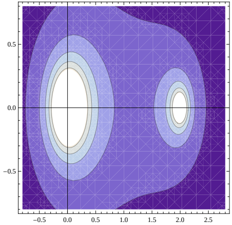
- Consider the map:

$$w = z^{\frac{1}{n}}$$



- Using conformal covariance:

$$\langle \Phi(z_1)\Phi(z_2)\rangle_{\mathcal{R}_n} = \left|\frac{dw_1}{dz_1}\right|^X \left|\frac{dw_2}{dz_2}\right|^X \frac{1}{|w_1 - w_2|^{2X}}$$
$$= \frac{1}{n^{2X}} \frac{(\rho_1\rho_2)^{(1-n)X}}{[\rho_1^2 + \rho_2^2 - 2\rho_1\rho_2 \cos\varphi]^X}$$



- Extra singularity at the branch point: $z \to 0$,
- Consistent with interpretation: $\Phi(z) \stackrel{z\to 0}{\longrightarrow} |z|^{\left(\frac{1}{n}-1\right)X}\Phi^{(n)}(0)$.
- Corollary: at the branch point scaling dimension of Φ is reduced: $\frac{X}{n}$

Usual corrections to scaling in QFT

- Usual corrections to scaling in a lattice QFT are due to irrelevant operators:

$$\mathcal{A} = \mathcal{A}_{scaling} + \lambda \int d^2x \Phi(x)$$

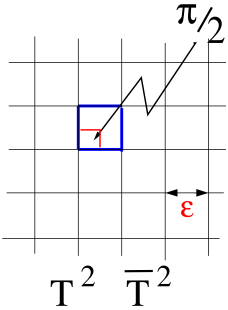
- There are two scales in the action $\mathcal{A}$
  1)correlation length ξ
  2)cut off (lattice) $\lambda = g\varepsilon^{-d+X_\Phi}$
- Free energy is then corrected by:

$$\delta F = \lambda \int d^2x \langle \Phi(x) \rangle_{\mathcal{A}_{scaling}} \propto \lambda \xi^{2-X_\Phi} = g \left(\frac{\xi}{\varepsilon} \right)^{2-X_\Phi}$$

- We expect then, for $l \gg \xi$:

$$F = A + B \left(\frac{\xi}{\varepsilon} \right)^{2-X_\Phi}$$

Ex: Ising



Unusual corrections to scaling for Reny entropies

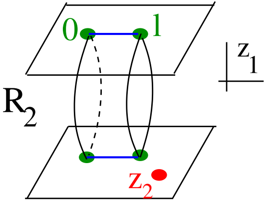
- After all $S_n \propto -(F_n - nF_1)$ and we computed at the fixed point with the conformal invariant action $\mathcal{A}_{CFT}$

- Same formalism: corrections to scaling (lattice dependent quantities) are given by irrelevant operators:

$$\mathcal{A} = \mathcal{A}_{CFT} + \lambda \int_{\mathcal{R}_n} \mathrm{d}^2 x \Phi(x); \qquad \lambda = g\varepsilon^{X_\Phi - 2}$$

- Perturbation theory in λ:

$$\delta F_n = \sum_{k=1}^{\infty} \frac{(-\lambda)^{k+1}}{k!} \int_{\mathcal{R}_n} \mathrm{d}^2 z_1 \int_{\mathcal{R}_n} \mathrm{d}^2 z_k \langle \Phi(z_1) \ldots \Phi(z_k) \rangle_{\mathcal{R}_n},$$

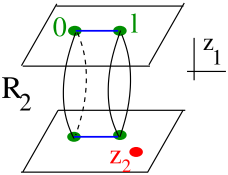


contains UV divergences regulated by a cut-off $\varepsilon^{-\alpha}$.

- But F_n is dimensionless and only other length in the theory is l:

$$\delta F_n \sim o\left(\frac{l}{\varepsilon}\right)^{\alpha}$$

Analysis of divergences: irrelevant operators



- First contribution to δF_n is:

$$\delta F_n = -\frac{g^2 \varepsilon^{2(x_\Phi - 2)}}{2} \int_{\mathcal{R}_n} d^2 z_1 \int_{\mathcal{R}_n} d^2 z_2 \langle \Phi(z_1) \Phi(z_2) \rangle_{\mathcal{R}_n}$$

$z_1 \longrightarrow z_2$

- Singularity is proportional to:

$$\text{Area}(\mathcal{R}_n)\varepsilon^{-2}$$

- Cancels in $F_n - nF_1$!
- Prefactor produces:

$$\delta S_n \propto \left(\frac{l}{\varepsilon}\right)^{2(2-x_\Phi)}$$

$z_1 \longrightarrow 0$ (or l)

- Two points functions goes like:

$$\langle \Phi(z_1) \Phi(z_2) \rangle_{\mathcal{R}_n} \sim \varepsilon^{\left(\frac{1}{n}-1\right)x_\Phi}$$

- Corrections to scalings are:

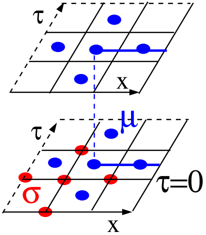
$$\delta S_n \propto \left(\frac{l}{\varepsilon}\right)^{-\left(\frac{1}{n}+1\right)x_\Phi + 2}, \quad \left(\frac{l}{\varepsilon}\right)^{-2\frac{x_\Phi}{n}}$$

- Observed numerically?

Relevant operators

- However nothing prevent the presence of relevant operators at the branch point, even flowing to a fixed point in the bulk!

- Example: Ising model



Near the branch points...

- Ising spin σ_i has 4 nn
- Ising disorder variable μ_i have $4 \times n$ nn
- Model cannot be critical near the branch point!

- We allow in the field theory action not only irrelevant perturbations but relevant operators localized on the branch points:

$$\mathcal{A} = \mathcal{A}_{CFT} + \sum_j \lambda_j \int d^2x \Phi_j(x) + \sum_{P,k} \lambda_k \Phi_k^{(n)}(P); \quad \dim \Phi_k^{(n)}(P) = \frac{X_\Phi}{n}$$

- Immediately gives a corrections to scaling of type:

$$\delta S_n \sim \left(\frac{l}{\varepsilon}\right)^{-\frac{2X_\Phi}{n}} \qquad X_\Phi < 2$$

Final comment: marginal case

- In the case exists a marginal (irrelevant) perturbation $X_\Phi = 2$

$$S_n\left(\frac{l}{\varepsilon}\right) = \frac{c_{eff}\left(\frac{l}{\varepsilon}\right)}{1+n^{-1}} \log\frac{l}{\varepsilon} \qquad \text{with:}$$

$$c_{eff}\left(\frac{l}{\varepsilon}\right) = c - \frac{1}{b^2\log^3\left(\frac{l}{\varepsilon}\right)} \qquad \text{b from: } \Phi\cdot\Phi = \mathbf{1} + b\Phi$$

- Question: It is possible to think $c_{eff}(l)$ as a C function? Actually not but...

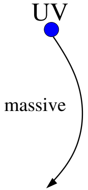
RK: [Casini-Huerta 2006]

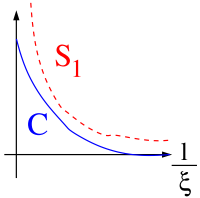
- Take the Von Neumman entropy $S_1(l)$,

$$C(l) \equiv l\frac{d}{dl}S_1(l)$$

is a C function:

1. Is monotonically decreasing with the $\frac{l}{\xi}$
2. $C(\frac{l}{\xi} \to 0)$ is the central charge





What I tried to tell you...

- The behaviour of entanglement entropies for $1+1$ dimensional systems e.g. critical spin chain is subjected to corrections

$$\left(\frac{l}{\varepsilon}\right)^{\alpha},$$

  due to the lattice.

- The exponent α can be computed within (Conformal) Field theory and we get an unusual correction of type:

$$\left(\frac{l}{\varepsilon}\right)^{-2\frac{X_{\Phi}}{n}} ; \quad X_{\Phi} < 2$$

- For the case of the XX spin chain ($\sim$ 2Ising) this is consistent with $X_{\Phi} = 1$ i.e. to a coupling to the energy at the branch points
- Of course other checks are needed: in particular modifying near the branch points the Hamiltonian, we can get rid of these corrections completely?

References

- Basic on Entanglement entropy in 1+1 dimension:
  - Calabrese-Cardy; Entanglement entropy and Quantum Field Theory; `arXiv: 0905.4013`; 2009
  - Casini-Huerta; A c theorem for entanglement entropy; `arXiv: cond-mat/0610375`; 2006
  - Castro Alvaredo-Doyon; Bipartite entanglement entropy in 1+1 massive QFT; `arXiv: 0906.2946`; 2009
- This Journal Club:
  - Cardy-Calabrese; Unusual corrections to scaling in Entanglement entropy; `arXiv: 1002.4353`; 2010
  - Calabrese et al.; Parity effects in the scaling block entanglement in gapless spin chains; `arXiv: 0911.4660`; 2009