Unusual corrections to scaling in entanglement entropy

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Summary

- Entanglement entropies in 1+1 QFT (review)
- Usual and unusual corrections to scaling in Entanglement entropies
Consider a many body state $|\psi\rangle$, i.e. the g.s. of a many body Hamiltonian $H$.

My simplest example is the g.s. of a spin chain Simplest because: $\mathcal{H} = \bigotimes_{i=1}^{N} \mathcal{H}_i$,

$$
|\psi\rangle = \sum_{i_1, \ldots, i_N = \pm} \sum_{d_B} \sum_{d_A} c_{i_1, \ldots, i_N} |i_1, \ldots, i_N\rangle = \sum_{j=1}^{d_B} \sum_{i=1}^{d_A} c_{ij} |\psi^A_i\rangle |\phi^B_j\rangle
$$

We allow manipulations on $\mathcal{H}_A (U(d_A))$ and $\mathcal{H}_B, (U(d_B))$ and ask: How much $|\psi\rangle$ is factorized? Useful quantity:

$$(\rho_A)_{ij} = (\text{Tr}_B |\psi\rangle \langle \psi|)_{ij} = \sum_{k=1}^{d_B} c_{ik} c_{kj}^* = (CC^\dagger)_{ij}$$
Entanglement entropies (again)

- After partitioning the Hilbert space $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$

$$|\psi\rangle = \sum_{i=1}^{d_A} \sum_{j=1}^{d_B} c_{ij} |\psi_i^A\rangle |\phi_j^B\rangle$$

- Diagonalizing the matrix $C$:

$$\psi_1, \ldots, \psi_{d_A} \rightarrow \phi_1, \ldots, \phi_{d_B}$$

- Trivial fact: $\lambda_i$ are eigenvalues of $\rho_A$ ($\rho_B$)

$$|\psi\rangle = \sum_{i=1}^{N_s} \sqrt{\lambda_i} |\psi'_i\rangle |\phi'_i\rangle$$ Schmidt decomposition

- Quantify how much $\lambda_i$ are spread inside $|\psi\rangle$:

$$S_n (|\psi\rangle) = \frac{1}{1 - n} \log \text{Tr} \rho_A^n$$
A question...

**Question:**
- Q: Given $c_{ij}$, randomly distributed on a sphere $d_A \times d_B$ how it is distributed the Von Neumann entropy $S_1 (|\psi\rangle)$?

**Answer:** [Lewenstein, Les Houches lectures 2009]
- If $d_A \ll d_B$:
  $$\langle S_1 (|\psi\rangle) \rangle \sim \log d_A - \frac{d_A}{2d_B} \quad \text{(volume law)},$$

  **Means:** A many body random state is typically maximally entangled (of course physical states are eigenstates of Hamiltonian...).
Replicas and Reny entropies

- Consider a $d = 1 + 1$ QFT at finite temperature $\frac{1}{\beta}$

$$\rho = \frac{1}{Z} \sum_{\text{states}} e^{-\beta E_n} |E_n\rangle \langle E_n| \xrightarrow{\beta \to \infty} |0\rangle \langle 0|$$

- Partition function can be represented as:

$$Z = \int \mathcal{D}\Phi e^{-\int_0^\beta d\tau \mathcal{L}_E[\phi]}$$

- The reduced density matrix is:

$$\rho_A (\phi_+ (x, 0^+); \phi_- (x, 0^-))$$

- In the limit $\beta \to \infty$:

$$\text{Tr} \rho_A^n = \frac{Z_n}{Z^n}, \quad \text{Partition function on a n-sheeted Riemann surface}$$
Scaling of the Reny entropies

In particular Reny entropies are free energies on Riemann surfaces:

\[ S_n \propto \log \text{Tr} \rho_A^n = -[F_n - nF_1] \]

Rk: Terms \( o(\varepsilon^{-2}) \) and \( o(\varepsilon^{-1}) \) cancel in \( S_n \) (volume and area). Indeed:

\[ S_n \sim A\varepsilon^{-d+2}, \quad \text{(Area law)} \]

For a 1 + 1 CFT:

\[ S_n = -\frac{1}{1 - n}(F_n - nF_1) = \frac{c}{6} \left( 1 + \frac{1}{n} \right) \log \frac{l}{\varepsilon} \]

Huge correlations due to massless particles (how? which?)
The proof for a 1+1 CFT

There are (at least) three equivalent ways to get the 1+1 CFT result:

\[ S_n = \frac{c}{6} \left( 1 + \frac{1}{n} \right) \log \frac{l}{\epsilon} \]

I chose this [Cardy, Les Houches lectures 2008]:

1. Apply a dilatation to the lattice:

\[ \epsilon \rightarrow (1 + \lambda)\epsilon \]

\[ \epsilon \frac{\partial F_n}{\partial \epsilon} = -\frac{1}{2\pi} \int_{\mathcal{R}_n} d^2z \langle \Theta(z, \bar{z}) \rangle_{\mathcal{R}_n} \]

2. However, \( \langle \Theta(z, \bar{z}) \rangle_{\mathcal{R}_n} = 0 \), but at the branch points where:

\[ \langle T(z) \rangle_{\mathcal{R}_n} \sim \frac{1}{z^2} \frac{c}{24} \left( 1 - \frac{1}{n^2} \right) \]

Result follows from:

\[ \frac{1}{2\pi} \int_{D} d^2z \langle \Theta \rangle_{\mathcal{R}_n} = -i \frac{n}{2\pi} \oint_{\gamma_D} dz z \langle T(z) \rangle_{\mathcal{R}_n} + \text{c.c.} \]
Corrections to the (Field Theory) scaling limit for Reny Entropies

- The CFT result holds in the scaling limit $l \gg \varepsilon$. However lattice effects can produce corrections:

$$S_n(l) = \frac{c}{6} \left(1 + \frac{1}{n}\right) \log \frac{l}{\varepsilon} + \sum_{\alpha < 0} A_\alpha \left(\frac{l}{\varepsilon}\right)^\alpha$$

- In [Calabrese et al. 2010] an analytic calculation of subleading term in $\frac{l}{\varepsilon}$ is reported for XX spin chain in zero magnetic field ($c = 1$).

$$S_n(l) = S_n^{\text{CFT}}(l) + (-1)^l f_n \left(\frac{l}{\varepsilon}\right)^{-\frac{2}{n}}$$

- That was checked numerically ($\alpha = n$):

- Can Field Theory predict this power?
- Rk: Actually XX Reny entropies are just twice as Ising!
Detour: Two points function on a Riemann surface

Consider the map:

\[ w = z^{\frac{1}{n}} \]

Using conformal covariance:

\[
\langle \Phi(z_1)\Phi(z_2) \rangle_{\mathcal{R}_n} = \left| \frac{dw_1}{dz_1} \right|^X \left| \frac{dw_2}{dz_2} \right|^X \frac{1}{|w_1 - w_2|^{2X}} \]

\[
= \frac{1}{n^{2X}} \frac{(\rho_1 \rho_2)^{(1-n)X}}{[\rho_1^2 + \rho_2^2 - 2\rho_1 \rho_2 \cos \varphi]^X} \]

Extra singularity at the branch point: \( z \to 0 \),

Consistent with interpretation: \( \Phi(z) \xrightarrow{z \to 0} |z|^{(\frac{1}{n} - 1)X} \Phi^{(n)}(0) \).

Corollary: at the branch point scaling dimension of \( \Phi \) is reduced: \( \frac{X}{n} \)
Usual corrections to scaling in QFT

- Usual corrections to scaling in a lattice QFT are due to irrelevant operators:
  \[ \mathcal{A} = \mathcal{A}_{\text{scaling}} + \lambda \int d^2 x \Phi(x) \]

- There are two scales in the action \( \mathcal{A} \):
  1) correlation length \( \xi \)
  2) cut off (lattice) \( \lambda = g \varepsilon^{-d+\chi} \)

- Free energy is then corrected by:
  \[ \delta F = \lambda \int d^2 x \langle \Phi(x) \rangle \mathcal{A}_{\text{scaling}} \propto \lambda \xi^{2-\chi} = g \left( \frac{\xi}{\varepsilon} \right)^{2-\chi} \]

- We expect then, for \( l \gg \xi \):
  \[ F = A + B \left( \frac{\xi}{\varepsilon} \right)^{2-\chi} \]

Ex: Ising

Ex: Ising

\[ \pi/2 \]

\[ \varepsilon \]

\[ T^2 \quad \bar{T}^2 \]
Unusual corrections to scaling for Reny entropies

- After all $S_n \propto -(F_n - nF_1)$ and we computed at the fixed point with the conformal invariant action $A_{CFT}$

- Same formalism: corrections to scaling (lattice dependent quantities) are given by irrelevant operators:

$$A = A_{CFT} + \lambda \int_{\mathcal{R}_n} d^2x \Phi(x); \quad \lambda = g\varepsilon^{\chi\Phi - 2}$$

- Perturbation theory in $\lambda$:

$$\delta F_n = \sum_{k=1}^{\infty} \frac{(-\lambda)^{k+1}}{k!} \int_{\mathcal{R}_n} d^2z_1 \int_{\mathcal{R}_n} d^2z_k \langle \Phi(z_1) \ldots \Phi(z_k) \rangle_{\mathcal{R}_n},$$

contains UV divergences regulated by a cut-off $\varepsilon^{-\alpha}$.

- But $F_n$ is dimensionless and only other length in the theory is $l$:

$$\delta F_n \sim o \left( \frac{l}{\varepsilon} \right)^{\alpha}$$
Analysis of divergences: irrelevant operators

- First contribution to $\delta F_n$ is:

$$\delta F_n = \frac{-g^2 \varepsilon^2 (x_\phi - 2)}{2} \int_{R_n} d^2 z_1 \int_{R_n} d^2 z_2 \langle \Phi(z_1)\Phi(z_2) \rangle_{R_n}$$

- **Singularity is proportional to:**
  
  $\text{Area}(R_n) \varepsilon^{-2}$

- **Cancels in $F_n - nF_1$!**

- **Prefactor produces:**

$$\delta S_n \propto \left( \frac{l}{\varepsilon} \right)^{2(2-x_\phi)}$$

- **Observed numerically?**

- **Two points functions goes like:**

$$\langle \Phi(z_1)\Phi(z_2) \rangle_{R_n} \sim \varepsilon^{\left( \frac{1}{n} - 1 \right)x_\phi}$$

- **Corrections to scalings are:**

$$\delta S_n \propto \left( \frac{l}{\varepsilon} \right)^{-\left( \frac{1}{n} + 1 \right)x_\phi + 2}, \quad \left( \frac{l}{\varepsilon} \right)^{-2\frac{x_\phi}{n}}$$
Relevant operators

- However nothing prevent the presence of relevant operators at the branch point, even flowing to a fixed point in the bulk!

- Example: Ising model

Near the branch points...

- Ising spin $\sigma_i$ has 4 nn
- Ising disorder variable $\mu_i$ have $4 \times n$ nn
- Model cannot be critical near the branch point!

- We allow in the field theory action not only irrelevant perturbations but relevant operators localized on the branch points:

$$\mathcal{A} = \mathcal{A}_{CFT} + \sum_j \lambda_j \int d^2 x \Phi_j(x) + \sum_{P, k} \lambda_k \Phi^{(n)}_k(P); \quad \text{dim} \Phi^{(n)}_k(P) = \frac{X_\Phi}{n}$$

- Immediately gives a corrections to scaling of type:

$$\delta S_n \sim \left( \frac{1}{\varepsilon} \right)^{-\frac{2X_\Phi}{n}} \quad X_\Phi < 2$$
Final comment: marginal case

- In the case exists a marginal (irrelevant) perturbation $X_\phi = 2$

$$S_n\left(\frac{l}{\varepsilon}\right) = \frac{c_{\text{eff}}\left(\frac{l}{\varepsilon}\right)}{1 + n^{-1}} \log \frac{l}{\varepsilon} \quad \text{with:}$$

$$c_{\text{eff}}\left(\frac{l}{\varepsilon}\right) = c - \frac{1}{b^2 \log^3 \left(\frac{l}{\varepsilon}\right)}$$

- Question: It is possible to think $c_{\text{eff}}(l)$ as a C function? Actually not but...

RK: [Casini-Huerta 2006]

- Take the Von Neumman entropy $S_1(l)$,

$$C(l) \equiv \frac{d}{dl} S_1(l)$$

is a C function:
1. Is monotonically decreasing with the $\frac{l}{\xi}$
2. $C\left(\frac{l}{\xi} \rightarrow 0\right)$ is the central charge
What I tried to tell you...

- The behaviour of entanglement entropies for 1 + 1 dimensional systems e.g. critical spin chain is subjected to corrections

\[ \left( \frac{l}{\varepsilon} \right)^\alpha, \]

due to the lattice.

- The exponent \( \alpha \) can be computed within (Conformal) Field theory and we get an unusual correction of type:

\[ \left( \frac{l}{\varepsilon} \right)^{-2\frac{X}{n}} \text{; } X_{\Phi} < 2 \]

- For the case of the \( XX \) spin chain (\( \sim 2\text{Ising} \)) this is consistent with \( X_{\Phi} = 1 \) i.e. to a coupling to the energy at the branch points

- Of course other checks are needed: in particular modifying near the branch points the Hamiltonian, we can get rid of these corrections completely?
References

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